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Oriented Circles in Non-Euclidean Space III.

By

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INTRODUCTION.

1. In this paper we shall discuss the congruences of oriented circles by the method which Fibbi[1] has used in treating the congruences of lines in non-euclidean space.

We shall define as a congruence of oriented circles a system whose Study's coordinates[2] are analytic functions of two independent parameters, say u and v. This is equivalent to supposing that our circles are determined by a sphere (x) and a plane (y) passing through its center, whose coordinates are analytic functions of two independent parameters u and v,

$$(1) \qquad \begin{aligned} x_i &= x_i\,(u, v), \\ y_i &= y_i\,(u, v), \end{aligned} \qquad (i=0, 1, 2, 3, 4).$$

Let (a) be the coordinates of the center of the sphere whose radius is r and (b) that of the plane of the circles, then we can put

$$x_i = \frac{a_i}{\sin\frac{r}{k}}, \qquad (i=0, 1, 2, 3).$$

1 Fibbi, 'I sistemi doppiamenete infiniti di raggi negli spazii di curvatura constante,' Annali della R. Scuola Normale Superiore, Pisa, 1891. Also see Coolidge, Non-Euclidean Geometry.

2 Nishiuchi and Kashiwagi, 'Oriented Circles in Non-Euclidean Space,' these Memoirs, 4, 273 (1920).

$$x_4 = ik\frac{\cos\frac{r}{k}}{\sin\frac{r}{k}},$$

(2)

$$y_0 = b_0, \qquad y_1 = b_1, \qquad y_2 = b_2, \qquad y_3 = b_3, \qquad y_4 = 0,$$

where

$$(aa) = k^2, \qquad (bb) = k^2, \qquad (ab) = 0,$$

and we have

$$(xx) = k^2, \qquad (yy) = k^2, \qquad (xy) = 0.$$

According to Kummer's classical method,[1] we shall write the following fundamental quadratic expression:

$$k^2(dxdx) - (ydx)^2 = \left\| \begin{matrix} y_0 & y_1 & y_2 & y_3 & y_4 \\ dx_0 & dx_1 & dx_2 & dx_3 & dx_4 \end{matrix} \right\|^2$$

$$= \boldsymbol{E}\, du^2 + 2\,\boldsymbol{F}\, dudv + \boldsymbol{G} dv^2,$$

(3)

$$k^2(dydy) - (xdy)^2 = \left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ dy_0 & dy_1 & dy_2 & dy_3 & dy_4 \end{matrix} \right\|^2$$

$$= \boldsymbol{E}'\, du^2 + 2\boldsymbol{F}'\, dudv + \boldsymbol{G}'\, dv^2,$$

$$k^2(dxdy) = \left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ dx_0 & dx_1 & dx_2 & dx_3 & dx_4 \end{matrix} \right\| \cdot \left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ dy_0 & dy_1 & dy_2 & dy_3 & dy_4 \end{matrix} \right\|$$

$$= \left\| \begin{matrix} y_0 & y_1 & y_2 & y_3 & y_4 \\ dy_0 & dy_1 & dy_2 & dy_3 & dy_4 \end{matrix} \right\| \cdot \left\| \begin{matrix} y_0 & y_1 & y_2 & y_3 & y_4 \\ dx_0 & dx_1 & dx_2 & dx_3 & dx_4 \end{matrix} \right\|$$

$$= \boldsymbol{e}\, du^2 + (\boldsymbol{f} + \boldsymbol{f}')\, dudv + \boldsymbol{g}\, dv^2.$$

More specifically

$$k^2\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial u}\right) - \left(y\frac{\partial x}{\partial u}\right)^2 = \frac{1}{\sin^2\frac{r}{k}}\left[k^2\left(\frac{\partial a}{\partial u}\frac{\partial a}{\partial u}\right)_r - \left(b\frac{\partial a}{\partial u}\right)^2\right]$$

$$= \boldsymbol{E},$$

[1] Kummer 'Allgemeine Théorie der geradlinigen Strahlensysteme,' Crelles Journal, **62**, (1860).

$$k^2\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right)-\left(y\frac{\partial x}{\partial u}\right)\left(y\frac{\partial x}{\partial v}\right)=\frac{1}{\sin^2\frac{r}{k}}\left[k^2\left(\frac{\partial a}{\partial u}\frac{\partial a}{\partial v}\right)_r-\left(b\frac{\partial a}{\partial u}\right)\left(b\frac{\partial a}{\partial v}\right)\right]$$

$$=\boldsymbol{F},$$

$$k^2\left(\frac{\partial x}{\partial v}\frac{\partial x}{\partial v}\right)-\left(y\frac{\partial x}{\partial v}\right)^2=\frac{1}{\sin^2\frac{r}{k}}\left[k^2\left(\frac{\partial a}{\partial v}\frac{\partial a}{\partial v}\right)_r-\left(b\frac{\partial a}{\partial v}\right)^2\right]$$

$$=\boldsymbol{G},$$

$$k^2\left(\frac{\partial y}{\partial u}\frac{\partial y}{\partial u}\right)-\left(x\frac{\partial y}{\partial u}\right)^2=k^2\left(\frac{\partial b}{\partial u}\frac{\partial b}{\partial u}\right)-\frac{1}{\sin^2\frac{r}{k}}\left(a\frac{\partial b}{\partial u}\right)^2 \tag{4}$$

$$=\boldsymbol{E}',$$

$$k^2\left(\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\right)-\left(x\frac{\partial y}{\partial u}\right)\left(x\frac{\partial y}{\partial v}\right)=k^2\left(\frac{\partial b}{\partial u}\frac{\partial b}{\partial v}\right)-\frac{1}{\sin^2\frac{r}{k}}\left(a\frac{\partial b}{\partial u}\right)\left(a\frac{\partial b}{\partial v}\right)$$

$$=\boldsymbol{F}',$$

$$k^2\left(\frac{\partial y}{\partial v}\frac{\partial y}{\partial v}\right)-\left(x\frac{\partial y}{\partial v}\right)^2=k^2\left(\frac{\partial b}{\partial v}\frac{\partial b}{\partial v}\right)-\frac{1}{\sin^2\frac{r}{k}}\left(a\frac{\partial b}{\partial v}\right)^2$$

$$=\boldsymbol{G}',$$

$$k^2\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\right)=\frac{k^2\sin\frac{r}{k}\left(\frac{\partial b}{\partial u}\frac{\partial b}{\partial u}\right)-k\cos\frac{r}{k}\left(a\frac{\partial b}{\partial u}\right)\frac{\partial r}{\partial u}}{\sin^2\frac{r}{k}}$$

$$=\boldsymbol{e},$$

$$k^2\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)=\frac{k^2\sin\frac{r}{k}\left(\frac{\partial a}{\partial u}\frac{\partial b}{\partial u}\right)-k\cos\frac{r}{k}\left(a\frac{\partial b}{\partial u}\right)\frac{\partial r}{\partial v}}{\sin^2\frac{r}{k}}$$

$$=\boldsymbol{f},$$

$$k^2\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\right)=\frac{k^2\sin\frac{r}{k}\left(\frac{\partial a}{\partial u}\frac{\partial b}{\partial v}\right)-k\cos\frac{r}{k}\left(a\frac{\partial b}{\partial v}\right)\frac{\partial r}{\partial u}}{\sin^2\frac{r}{k}}$$

$$=\boldsymbol{f}',$$

$$k^2\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial v}\right)=\frac{k^2\sin\frac{r}{k}\left(\frac{\partial a}{\partial v}\frac{\partial b}{\partial v}\right)-k\cos\frac{r}{k}\left(a\frac{\partial b}{\partial v}\right)\frac{\partial r}{\partial v}}{\sin^2\frac{r}{k}}$$

$$=g,$$

(5)

$$EG-F^2=\left\|\begin{array}{ccccc} y_0 & y_1 & y_2 & y_3 & y_4 \\ x_0 & x_1 & x_2 & x_3 & x_4 \\ \frac{\partial x_0}{\partial u} & \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} & \frac{\partial x_4}{\partial u} \\ \frac{\partial x_0}{\partial v} & \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} & \frac{\partial x_4}{\partial v} \end{array}\right\|^2,$$

$$E'G'-F'^2=\left\|\begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \\ \frac{\partial y_0}{\partial u} & \frac{\partial y_1}{\partial u} & \frac{\partial y_2}{\partial u} & \frac{\partial y_3}{\partial u} & \frac{\partial y_4}{\partial u} \\ \frac{\partial y_0}{\partial v} & \frac{\partial y_1}{\partial v} & \frac{\partial y_2}{\partial v} & \frac{\partial y_3}{\partial v} & \frac{\partial y_4}{\partial v} \end{array}\right\|^2,$$

where

$$\left(\frac{\partial a}{\partial u}\frac{\partial a}{\partial u}\right)_r\equiv\left(\frac{\partial a_0}{\partial u}\right)^2+\left(\frac{\partial a_1}{\partial u}\right)^2+\left(\frac{\partial a_2}{\partial u}\right)^2+\left(\frac{\partial a_3}{\partial u}\right)^2-\left(\frac{\partial r}{\partial u}\right)^2.$$

And since

$$\left\|\begin{array}{cccccc} x_0 & x_1 & x_2 & x_3 & x_4 & 0 \\ y_0 & y_1 & \cdots\cdots & & y_4 & 0 \\ \frac{\partial x_0}{\partial u} & \frac{\partial x_1}{\partial u} & \cdots\cdots & & \frac{\partial x_4}{\partial u} & 0 \\ \frac{\partial x_0}{\partial v} & \frac{\partial x_1}{\partial v} & \cdots\cdots & & \frac{\partial x_4}{\partial v} & 0 \\ \frac{\partial y_0}{\partial u} & \frac{\partial y_1}{\partial u} & \cdots\cdots & & \frac{\partial y_4}{\partial u} & 0 \\ \frac{\partial y_0}{\partial v} & \frac{\partial y_1}{\partial v} & \cdots\cdots & & \frac{\partial y_4}{\partial v} & 0 \end{array}\right\|=0,$$

we have

$$(6)\qquad \begin{vmatrix} \boldsymbol{E} & \boldsymbol{F} & \boldsymbol{e} & \boldsymbol{f}' \\ \boldsymbol{F} & \boldsymbol{G} & \boldsymbol{f} & \boldsymbol{g} \\ \boldsymbol{e} & \boldsymbol{f} & \boldsymbol{E}' & \boldsymbol{F}' \\ \boldsymbol{f}' & \boldsymbol{g} & \boldsymbol{F}' & \boldsymbol{G}' \end{vmatrix} = 0.$$

Putting $r=\frac{\pi}{2}k$ in these expressions, we have

$$\boldsymbol{E}_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial a}{\partial u}\frac{\partial a}{\partial u}\right) - \left(b\frac{\partial a}{\partial u}\right)^2 \equiv E,$$

$$\boldsymbol{F}_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial a}{\partial u}\frac{\partial a}{\partial v}\right) - \left(b\frac{\partial a}{\partial u}\right)\left(b\frac{\partial a}{\partial v}\right) \equiv F,$$

$$\boldsymbol{G}_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial a}{\partial v}\frac{\partial a}{\partial v}\right) - \left(b\frac{\partial a}{\partial v}\right)^2 \equiv G,$$

$$\boldsymbol{E}'_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial b}{\partial u}\frac{\partial b}{\partial u}\right) - \left(a\frac{\partial b}{\partial u}\right)^2 \equiv E',$$

$$(7)\qquad \boldsymbol{F}'_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial b}{\partial u}\frac{\partial b}{\partial v}\right) - \left(a\frac{\partial b}{\partial u}\right)\left(a\frac{\partial b}{\partial v}\right) \equiv F',$$

$$\boldsymbol{G}'_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial b}{\partial v}\frac{\partial b}{\partial v}\right) - \left(a\frac{\partial b}{\partial v}\right)^2 \equiv G',$$

$$\boldsymbol{e}_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial a}{\partial u}\frac{\partial b}{\partial u}\right) \equiv e,$$

$$\boldsymbol{f}_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial a}{\partial v}\frac{\partial b}{\partial u}\right) \equiv f,$$

$$\boldsymbol{f}'_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial a}{\partial u}\frac{\partial a}{\partial v}\right) \equiv f',$$

$$\boldsymbol{g}_{r=\frac{\pi}{2}k} = k^2\left(\frac{\partial a}{\partial v}\frac{\partial b}{\partial v}\right) \equiv g,$$

$$EG-F^2=\left|\begin{array}{cc} y & x \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\end{array}\right|^2,$$

$$E'G'-F'^2=\left|\begin{array}{cc} x & y \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{array}\right|^2,$$

$$(EG-F^2)\ (E'G'-F'^2)=(eg-ff')^2.$$

and

$$(8)\qquad \left|\begin{array}{ccc} E & F & e \\ F & G & f \\ e & f & E'\end{array}\right| = \left|\begin{array}{ccc} E & F & f' \\ F & G & g \\ f' & g & G'\end{array}\right| = \left|\begin{array}{ccc} E & F & e \\ F & G & f \\ f' & g & F'\end{array}\right| = 0,$$

$$\left|\begin{array}{ccc} E & e & f \\ e & E' & F' \\ f' & F' & G'\end{array}\right| = \left|\begin{array}{ccc} G & f & g \\ f & E' & F' \\ g & F' & G'\end{array}\right| = \left|\begin{array}{ccc} F & f & G \\ e & E' & F' \\ f' & F' & G'\end{array}\right| = 0.$$

These are the corresponding expressions for the space axes of our circles of congruence.

LIMITING SPHERE.

2. In a previous memoir,[1] we have shown that two circles which are not paratactic have two common perpendiculars cospherical orthogonal to both. Let us find these for a circle of our congruence and an adjacent circle. The coordinates of the sphere through our circle may be written $\cos\phi(x)+\sin\phi(y)$, $-\sin\phi(x)+\cos\phi(y)$ while those of the sphere through the adjacent one will be $\lambda(x+dx)+\mu(y+dy)$, $-\mu(x+dx)+\lambda(y+dy)$. Then the conditions will be

$$\lambda\Big[[k^2+(xdx)]\sin\phi-(ydx)\cos\phi\Big]+\mu\Big[(xdy)\sin\phi-[k^2+(ydy)]\cos\phi\Big]=0,$$
$$-\lambda\Big[[k^2+(ydy)]\sin\phi+(xdy)\cos\phi\Big]+\mu\Big[(ydx)\sin\phi+[k^2+(xdx)]\cos\phi\Big]=0.$$

When this is simplified in view of the identities

$$(xdx)=-\tfrac{1}{2}(dxdx),\quad (ydy)=-\tfrac{1}{2}(dydy),$$
$$(xdy)+(ydx)=-(dxdy),$$

[1] Mem. Coll. Sci. Kyoto, **5**, 383 (1922).

and eliminating λ and μ, we have

$$(9)\quad \begin{vmatrix} [k^2-\frac{1}{2}(dxdx)]\sin\phi-(ydx)\cos\phi & (xdy)\sin\phi-[k^2-\frac{1}{2}(dydy)]\cos\phi \\ -[k^2-\frac{1}{2}(dydy)]\sin\phi-(xdy)\cos\phi & (ydx)\sin\phi+[k^2-\frac{1}{2}(dxdx)]\cos\phi \end{vmatrix}=0.$$

Casting aside infinitesimals above the second order

$$(10)\quad \begin{aligned}&[\boldsymbol{e}du^2+(\boldsymbol{f}+\boldsymbol{f}')dudv+\boldsymbol{g}dv^2](\sin^2\phi-\cos^2\phi)\\&+[(\boldsymbol{E}-\boldsymbol{E}')du^2+2(\boldsymbol{F}-\boldsymbol{F}')dudv+(\boldsymbol{G}-\boldsymbol{G}')dv^2]\sin\phi\cos\phi=0.\end{aligned}$$

In order to discuss this equation we write two others:

$$(11)\quad \begin{aligned}\boldsymbol{e}du^2+(\boldsymbol{f}+\boldsymbol{f}')dudv+\boldsymbol{g}dv^2&=0,\\(\boldsymbol{E}-\boldsymbol{E}')du^2+2(\boldsymbol{F}-\boldsymbol{F}')dudv+(\boldsymbol{G}-\boldsymbol{G}')dv^2&=0,\end{aligned}$$

i.e.

$$(11)\quad \begin{aligned}(dxdy)&=0,\\(dxdx)-(dydy)&=0.\end{aligned}$$

(1°) When they have no common root, we shall say that the congruence is *general.*

(2°) When they have one common root, let the foci of our circle be (α) and (β). We may write

$$\begin{aligned}\alpha_i&=x_i+iy_i,\\\beta_i&=x_i-iy_i,\end{aligned}$$

or

$$\begin{aligned}x_i&=\tfrac{1}{2}(\alpha_i+\beta_i),\\y_i&=\tfrac{i}{2}(\beta_i-\alpha_i),\quad (i=0,\ 1,\ 2,\ 3,\ 4)\end{aligned}$$

and our differential equation will be

$$(d\alpha d\alpha)=(d\beta d\beta)=0.$$

Hence we can easily see that each circle is paratactic or tangent to one adjacent circle.[1]

(3°) When they are equivalent to one another, then

$$(12)\quad \frac{\boldsymbol{E}-\boldsymbol{E}'}{\boldsymbol{e}}=\frac{2(\boldsymbol{F}-\boldsymbol{F}')}{\boldsymbol{f}+\boldsymbol{f}'}=\frac{\boldsymbol{G}-\boldsymbol{G}'}{\boldsymbol{g}}$$

And the surfaces of foci are conformally related, and the congruence shall be called *conformal.* Such a case, we shall discuss later on.

[1] Kashiwagi, loc. cit.

Let us begin with the general congruence. If $du : dv$ be given we usually get a unique value for $\tan 2\phi$, that is to say, two mutually orthogonal spheres. On the other hand, if ϕ be given, i.e., if two mutually orthogonal spheres through our circle, be given, we have a quadratic equation in $du : dv$, so that on each sphere through our circle will lie two circles orthogonal thereto, orthogonal and cospherical also to an adjacent of congruence.

These two circles will fall together if the equation (10) in $du : dv$ has equal root, i.e. if

$$(13)\quad \begin{vmatrix} e(\tan^2\phi-1)+(E-E')\tan\phi & \frac{f+f'}{2}(\tan^2\phi-1)+(F-F')\tan\phi \\ \frac{f+f'}{2}(\tan^2\phi-1)+(F-F')\tan\phi & g(\tan^2\phi-1)+(G-G')\tan\phi \end{vmatrix}=0.$$

This equation is unaltered if we replace ϕ by $\frac{\pi}{2}+\phi$.

Theorem. In a general congruence, through each circle will pass two pairs of mutually orthogonal spheres, on each of which there is but a single circle orthagonal to the given circle, cospherical and orthogonal to an adjacent circle of the congruence.

These spheres shall be called the *limiting spheres* through the circles. They correspond to maximum and minimum values for ϕ in (10).

FOCAL CONGRUENCE.

3. In a previous memoir,[1] we have proved that in a general congruence of circles, every circle meets four other consecutive circles; it intersects each of the circles at a single point, the four points being the focal points of the circle: and have discussed such a congruence of circles that two of the four focal points of a circle shall lie on one consecutive circle, and the other two lie on another consecutive circle.

We shall call such a congruence *be focal.*

If a circle intersect with the consecutive circle at two points, they are cospherical and the conditions for that is[2]

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1 Kashiwagi, Mem. Coll. Sci., **4**, 233 (1922).

2 Nishiuchi and Kashiwagi, loc. cit.

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ da_0 & da_1 & da_2 & da_3 \\ db_0 & db_1 & db_2 & db_3 \end{vmatrix} = 0,$$

and all the determinants of the matrix

$$\begin{vmatrix} a_0\sin\frac{r}{k}\frac{dr}{k}+da_0\cos\frac{r}{k} & a_1\sin\frac{r}{k}\frac{dr}{k}+da_1\cos\frac{r}{k}\ldots a_3\sin\frac{r}{k}\frac{dr}{k}+da_3\cos\frac{r}{k} \\ b_0 & b_1 \quad \ldots \quad b_3 \\ db_0 & db_1 \quad \ldots \quad db_3 \end{vmatrix}$$

are equal to zero. Or

$$A_i du^2+(B_i+B_i')dudv+C_i dv^2=0, \qquad (i=0,\ 1,\ 2,\ 3)$$

where

$$A_i=\frac{\partial}{\partial t_i}\begin{vmatrix} t & \frac{a}{k}\sin\frac{r}{k}\frac{\partial r}{\partial u}+\frac{\partial a}{\partial u}\cos\frac{r}{k} & b & \frac{\partial b}{\partial u} \end{vmatrix},$$

$$B_i=\frac{\partial}{\partial t_i}\begin{vmatrix} t & \frac{a}{k}\sin\frac{r}{k}\frac{\partial r}{\partial u}+\frac{\partial a}{\partial u}\cos\frac{r}{k} & b & \frac{\partial b}{\partial v} \end{vmatrix},$$

$$B_i'=\frac{\partial}{\partial t_i}\begin{vmatrix} t & \frac{a}{k}\sin\frac{r}{k}\frac{\partial r}{\partial v}+\frac{\partial a}{\partial v}\cos\frac{r}{k} & b & \frac{\partial b}{\partial u} \end{vmatrix},$$

$$C_i=\frac{\partial}{\partial t_i}\begin{vmatrix} t & \frac{a}{k}\sin\frac{r}{k}\frac{\partial r}{\partial v}+\frac{\partial a}{\partial v}\cos\frac{r}{k} & b & \frac{\partial b}{\partial v} \end{vmatrix},$$

$$(i=0,\ 1,\ 2,\ 3).$$

If the congruence is focal, these four equations are all equivalent to one another, that is

$$\frac{A_0}{C_0}=\frac{A_1}{C_1}=\frac{A_2}{C_2}=\frac{A_3}{C_3},$$

$$\frac{A_0}{B_0+B_0'}=\frac{A_1}{B_1+B_1'}=\frac{A_2}{B_2+B_2'}=\frac{A_3}{B_3+B_3'}.$$

Hence we have the conditions for the focal congruence

$$(14)\qquad \begin{vmatrix} \frac{a}{k}\sin\frac{r}{k}\frac{\partial r}{\partial u}+\frac{\partial a}{\partial u}\cos\frac{r}{k} & \frac{a}{k}\sin\frac{r}{k}\frac{\partial r}{\partial v}+\frac{\partial a}{\partial v}\cos\frac{r}{k} & b\,\frac{\partial b}{\partial u} \end{vmatrix}=0,$$

$$\begin{vmatrix} \frac{a}{k}\sin\frac{r}{k}\frac{\partial r}{\partial u}+\frac{\partial a}{\partial u}\cos\frac{r}{k} & \frac{a}{k}\sin\frac{r}{k}\frac{\partial r}{\partial v}+\frac{\partial a}{\partial v}\cos\frac{r}{k} & b\,\frac{\partial b}{\partial v} \end{vmatrix}=0,$$

i.e.

$$(15)\qquad \begin{vmatrix} \frac{\partial y}{\partial u} & x & y & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix}=0,$$

$$\begin{vmatrix} \frac{\partial y}{\partial v} & x & y & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix}=0.$$

Squaring, we get

$$(16)\qquad \begin{vmatrix} E & F & e \\ F & G & f \\ e & f & E' \end{vmatrix}=\begin{vmatrix} E & F & f' \\ F & G & g \\ f' & g & G' \end{vmatrix}=0,$$

Theorem. The necessary and sufficient conditions that a congruence should be focal are that

$$(16)\qquad \begin{vmatrix} E & F & e \\ F & G & f \\ e & f & E' \end{vmatrix},\quad \begin{vmatrix} E & F & f' \\ F & G & g \\ f' & g & G' \end{vmatrix}$$

are equal to zero.

Hence from the relation (6), when the congruence is focal we have

$$\begin{vmatrix} E & F & e \\ F & G & f \\ e & f & E' \end{vmatrix}=\begin{vmatrix} E & F & f' \\ F & G & g \\ f' & g & G' \end{vmatrix}=\begin{vmatrix} E & F & f' \\ F & G & f \\ f' & g & F' \end{vmatrix}=0,$$

and

$$\begin{vmatrix} E & e & f \\ e & E' & F' \\ f' & F' & g' \end{vmatrix}=\begin{vmatrix} G & f & g \\ f & E' & F' \\ g & F' & g' \end{vmatrix}=\begin{vmatrix} F & f & g \\ e & E & F \\ f' & F & G \end{vmatrix}=0.$$

4. From a previous discussion,[1] we see that the coordinates of a point of our circle are given by

[1] Kashiwagi, loc. cit.

$$x_i = a_i \cos\frac{r}{k} + (\xi_i \cos\theta + \eta_i \sin\theta)\sin\frac{r}{k},$$
$$(i = 0,\ 1,\ 2,\ 3)$$

where (ξ), (η) are two mutually orthogonal points on its plane axis and θ is the angle between the line determined by the points (a), (ξ) and (a), (x). And we also see that in a focal congruence the two values of θ which give two focal points upon the circle u=const. and v=const. are given by the equations

$$\left(a\frac{\partial\xi}{\partial v}\right)\cos\theta + \left(a\frac{\partial\eta}{\partial v}\right)\sin\theta = k\frac{\partial r}{\partial v}, \tag{17}$$

and

$$\left(a\frac{\partial\xi}{\partial u}\right)\cos\theta + \left(a\frac{\partial\eta}{\partial u}\right)\sin\theta = k\frac{\partial r}{\partial u}. \tag{17}$$

5. If a sphere (z) be orthogonal to all the circles of a congruence, then

$$(xz) = \left(\frac{\partial x}{\partial u}z\right) = \left(\frac{\partial x}{\partial v}z\right) = 0,$$

$$(yz) = \left(\frac{\partial y}{\partial u}z\right) = \left(\frac{\partial y}{\partial v}z\right) = 0.$$

i.e. all the determinants of the matrix

$$\left\|\begin{array}{cccc} x_0 & x_1 & \cdots\cdots & x_4 \\ y_0 & y_1 & \cdots\cdots & y_4 \\ \frac{\partial x_0}{\partial u} & \frac{\partial x_1}{\partial u} & \cdots\cdots & \frac{\partial x_4}{\partial u} \\ \frac{\partial x_0}{\partial v} & \frac{\partial x_1}{\partial v} & \cdots\cdots & \frac{\partial x_4}{\partial v} \\ \frac{\partial y_0}{\partial u} & \frac{\partial y_1}{\partial u} & \cdots\cdots & \frac{\partial y_4}{\partial u} \\ \frac{\partial y_0}{\partial v} & \frac{\partial y_1}{\partial v} & \cdots\cdots & \frac{\partial y_4}{\partial v} \end{array}\right\|$$

are equal to zero. And we have the theorem:

Theorem. If all the circles of a congruence be orthogonal to a sphere, the congruence is focal and the planes of the circles pass through the center of the sphere.

6. We shall call the *focal sphere*, the sphere through a circle which contains the adjacent circles of the congruence.

If the focal sphere be

$$(x\cos\phi + y\sin\phi),$$

we shall have

$$x_i\cos\phi + y_i\sin\phi = (x_i + dx_i)\cos(\phi + d\phi) + (y_i + dy_i)\sin(\phi + d\phi),$$

$$(dx_i\cos\phi + dy_i\sin\phi) - (x_i\sin\phi - y_i\cos\phi)d\phi = 0,$$

$$d\phi = (xdy) = -(ydx),$$

$$\left[\left(\frac{\partial x_i}{\partial u}\cos\phi + \frac{\partial y_i}{\partial u}\sin\phi\right) - \left(x_i\sin\phi - y_i\cos\phi\right)\left(x\frac{\partial y}{\partial u}\right)\right]du$$

$$+\left[\left(\frac{\partial x_i}{\partial v}\cos\phi + \frac{\partial y_i}{\partial v}\sin\phi\right) - \left(x_i\sin\phi - y_i\cos\phi\right)\left(x\frac{\partial y}{\partial v}\right)\right]dv = 0.$$

$$(i=0,\ 1,\ 2,\ 3,\ 4)$$

Multiplying through by $\frac{\partial y_i}{\partial u}$ and adding, then multiplying through by $\frac{\partial y_i}{\partial v}$ and adding again.

$$(18)\qquad \begin{aligned} &[e\,du + f\,dv]\cos\phi + [E'du + F'dv]\sin\phi = 0,\\ &[f'du + g\,dv]\cos\phi + [F'du + G'dv]\sin\phi = 0. \end{aligned}$$

Similarly

$$(19)\qquad \begin{aligned} &[e\,du + f'dv]\sin\phi + [E\,du + F\,dv]\cos\phi = 0,\\ &[f\,du + g\,dv]\sin\phi + [F\,du + G\,dv]\cos\phi = 0. \end{aligned}$$

Eliminating ϕ,

$$(20)\qquad \begin{vmatrix} e\,du + f\,dv & E'du + F'dv\\ f'du + g\,dv & F'du + G'dv \end{vmatrix} = 0,$$

$$\begin{vmatrix} e\,du + f'dv & E\,du + F\,dv\\ f\,du + g\,dv & F\,du + G\,dv \end{vmatrix} = 0.$$

Eliminating $du : dv$,

$$(21)\qquad \begin{vmatrix} \boldsymbol{E}'\tan\phi+\boldsymbol{e} & \boldsymbol{F}'\tan\phi+\boldsymbol{f} \\ \boldsymbol{F}'\tan\phi+\boldsymbol{f}' & \boldsymbol{G}'\tan\phi+\boldsymbol{g} \end{vmatrix} = 0,$$

$$\begin{vmatrix} \boldsymbol{e}\tan\phi+\boldsymbol{E} & \boldsymbol{f}'\tan\phi+\boldsymbol{F} \\ \boldsymbol{f}\tan\phi+\boldsymbol{F} & \boldsymbol{g}\tan\phi+\boldsymbol{G} \end{vmatrix} = 0.$$

Subtracting one of these equations from the other

$$(22)\qquad \begin{aligned}[(\boldsymbol{eg}-\boldsymbol{ff}')-(\boldsymbol{E}'\boldsymbol{G}'-\boldsymbol{F}'^2)]\tan^2\phi+[(\boldsymbol{E}-\boldsymbol{E}')\boldsymbol{g}-(\boldsymbol{F}-\boldsymbol{F}')(\boldsymbol{f}+\boldsymbol{f}')\\ +(\boldsymbol{G}-\boldsymbol{G}')\boldsymbol{e}]\tan\phi+[(\boldsymbol{EG}-\boldsymbol{F}^2)-(\boldsymbol{eg}-\boldsymbol{ff}')]=0.\end{aligned}$$

In order that a congruence should be focal, it is necessary that the two equations (20) should be equivalent, and the same for the two equations (21). We see at once that the middle coefficients are identical in (13) and (21), and these will vanish when, and only when, the corresponding pairs of spheres make equal angles with (x) or (y).

It should, further, be noted that the focal spheres are mutually orthogonal when

$$(23)\qquad \begin{vmatrix} \boldsymbol{E} & \boldsymbol{F} \\ \boldsymbol{F} & \boldsymbol{G} \end{vmatrix} = \begin{vmatrix} \boldsymbol{E}' & \boldsymbol{F}' \\ \boldsymbol{F}' & \boldsymbol{G}' \end{vmatrix} = -\begin{vmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{f}' & \boldsymbol{g} \end{vmatrix}.$$

7. When our congruence of circles is focal, we may take $u=$const. and $v=$const. to be two consecutive circles each of which meets our circle at two points and we can find four quantities λ, μ and λ', μ' such that

$$(24)\qquad \begin{aligned}\frac{\partial b_i}{\partial u} &= \lambda\frac{a_i}{k}\sin\frac{r}{k}\frac{\partial r}{\partial u}+\lambda\frac{\partial a_i}{\partial u}\cos\frac{r}{k}+\mu b_i,\\ \frac{\partial b_i}{\partial v} &= \lambda'\frac{a_i}{k}\sin\frac{r}{k}\frac{\partial r}{\partial v}+\lambda'\frac{\partial a_i}{\partial v}\cos\frac{r}{k}+\mu' b_i.\end{aligned}$$

Multiplying through a_i, b_i and adding

$$\left(a\frac{\partial b}{\partial u}\right)=\lambda k\sin\frac{r}{k}\frac{\partial r}{\partial u},\qquad \left(a\frac{\partial b}{\partial v}\right)=\lambda' k\sin\frac{r}{k}\frac{\partial r}{\partial v},$$

$$-\lambda\left(b\frac{\partial a}{\partial u}\right)\cos\frac{r}{k}=\mu k^2,\qquad -\lambda'\left(b\frac{\partial a}{\partial v}\right)\cos\frac{r}{k}=\mu' k^2$$

Hence we have

$$(25)\qquad \begin{aligned} \frac{\partial b_i}{\partial u} &= a_i \frac{\mu}{\lambda \cos\frac{r}{k}} + \mu b_i + \lambda \cos\frac{r}{k}\frac{\partial a_i}{\partial u}, \\ \frac{\partial b_i}{\partial v} &= a_i \frac{\mu'}{\lambda' \cos\frac{r}{k}} + \mu' b_i + \lambda' \cos\frac{r}{k}\frac{\partial a_i}{\partial v}. \end{aligned}$$

$$(i=0,\ 1,\ 2,\ 3)$$

Or

$$(26)\qquad \begin{aligned} \frac{\partial y_i}{\partial u} &= x_i \frac{\mu}{\lambda \sin\frac{r}{k}\cos\frac{r}{k}} + \mu y_i + \lambda \sin\frac{r}{k}\cos\frac{r}{k}\frac{\partial x_i}{\partial u}, \\ \frac{\partial y_i}{\partial v} &= x_i \frac{\mu'}{\lambda' \sin\frac{r}{k}\cos\frac{r}{k}} + \mu' y_i + \lambda' \sin\frac{r}{k}\cos\frac{r}{k}\frac{\partial x_i}{\partial v}, \end{aligned}$$

$$(i=0,\ 1,\ 2,\ 3,\ 4)$$

and conversely.

Such parameters shall be called *focal parameters.*

Substituting these in (4) and (7), we have, for the congruence of the space axes of our circles

$$(27)\qquad \begin{aligned} e &= \lambda \cos\frac{r}{k} E', & f &= \lambda \cos\frac{r}{k} F, \\ f' &= \lambda' \cos\frac{r}{k} F, & g &= \lambda' \cos\frac{r}{k} G, \\ E' &= \lambda \cos\frac{r}{k} e, & G &= \lambda' \cos\frac{r}{k} g, \end{aligned}$$

$$F' = \lambda \cos\frac{r}{k} f = \lambda' \cos\frac{r}{k} f'.$$

and for our congruence of circles

$$(28)\qquad \begin{aligned} \boldsymbol{e} &= \lambda \sin\frac{r}{k}\cos\frac{r}{k}\boldsymbol{E}, & \boldsymbol{f} &= \lambda \sin\frac{r}{k}\cos\frac{r}{k}\boldsymbol{F}, \\ \boldsymbol{f}' &= \lambda' \sin\frac{r}{k}\cos\frac{r}{k}\boldsymbol{F}, & \boldsymbol{g} &= \lambda' \sin\frac{r}{k}\cos\frac{r}{k}\boldsymbol{G}, \\ \boldsymbol{E}' &= \lambda \sin\frac{r}{k}\cos\frac{r}{k}\boldsymbol{e}, & \boldsymbol{G}' &= \lambda' \sin\frac{r}{k}\cos\frac{r}{k}\boldsymbol{g}, \end{aligned}$$

$$\boldsymbol{F}' = \lambda' \sin\frac{r}{k}\cos\frac{r}{k}\boldsymbol{f} = \lambda \sin\frac{r}{k}\cos\frac{r}{k}\boldsymbol{f}'.$$

Putting these in the equation (20), we have

$$\left(\lambda \sin\frac{r}{k}\cos\frac{r}{k}\tan\phi + 1\right)\left(\lambda' \sin\frac{r}{k}\cos\frac{r}{k}\tan\phi + 1\right) = 0,$$

and the focal spheres are

$$(29) \qquad \lambda \sin\frac{r}{k}\cos\frac{r}{k}(x) - (y), \quad \lambda' \sin\frac{r}{k}\cos\frac{r}{k}(x) - (y),$$

or

$$(30) \qquad \begin{aligned} &\lambda \cos\frac{r}{k}a_0 - b_0, \quad \lambda \cos\frac{r}{k}a_1 - b_1, \ \ldots\ldots\ldots, \quad i\lambda \cos^2\frac{r}{k}, \\ &\lambda' \cos\frac{r}{k}a_0 - b_0, \quad \lambda' \cos\frac{r}{k}a_1 - b_1, \ \ldots\ldots\ldots, \quad i\lambda' \cos^2\frac{r}{k}. \end{aligned}$$

And the equations[1] for the focal points of the space axes of our congruence of circles

$$\begin{vmatrix} E'\tan\phi + e & F'\tan\phi + f \\ F'\tan\phi + f' & G'\tan\phi + g \end{vmatrix} = 0,$$

$$\begin{vmatrix} e\tan\phi + E, & f'\tan\phi + F \\ f\tan\phi + F & g\tan\phi + G \end{vmatrix} = 0$$

will be

$$\left(\lambda \cos\frac{r}{k}\tan\phi + 1\right)\left(\lambda' \cos\frac{r}{k}\tan\phi + 1\right) = 0.$$

and the focal points are

$$(36) \qquad \begin{aligned} &\lambda \cos\frac{r}{k}a_0 - b_0, \quad \lambda \cos\frac{r}{k}a_1 - b_1, \ \ldots\ldots\ldots, \quad \lambda \cos\frac{r}{k}a_3 - b_3, \\ &\lambda' \cos\frac{r}{k}a_0 - b_0, \quad \lambda' \cos\frac{r}{k}a_1 - b_1, \ \ldots\ldots\ldots, \quad \lambda' \cos\frac{r}{k}a_3 - b_3. \end{aligned}$$

Theorem. When a congruence is focal, the centers of the focal spheres will coincide with the focal points of the congruence of space axes of our circles.

Theorem. When a congruence is focal, the centers of gravity of the centers of the focal spheres are identical with those of two pairs of limiting points of the space axes af the congruence.

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[1] Fibbi, loc. cit.; Coolidge, loc. cit.

8. By a previous memoir, the necessary and sufficient condition that a congruence of circles may form a cyclical system is that, for all values of u, v and θ,

$$A+B\sin\theta+C\cos\theta=0,$$

must satisfy, where

$$A=\cos^2\frac{r}{k}\left[\left(\xi\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial a}{\partial v}\right)-\left(\eta\frac{\partial a}{\partial u}\right)\left(\xi\frac{\partial a}{\partial v}\right)\right]$$

$$+k^2\sin^2\frac{r}{k}\left[\frac{\partial}{\partial v}\left(\eta\frac{\partial\xi}{\partial u}\right)-\frac{\partial}{\partial u}\left(\eta\frac{\partial\xi}{\partial v}\right)\right],$$

$$B=\sin\frac{r}{k}\cos\frac{r}{k}\left[\left(\eta\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial\xi}{\partial v}\right)-\left(\eta\frac{\partial a}{\partial v}\right)\left(\eta\frac{\partial\xi}{\partial u}\right)\right.$$

$$\left.+k^2\left\{\frac{\partial}{\partial u}\left(\xi\frac{\partial a}{\partial v}\right)-\frac{\partial}{\partial v}\left(\xi\frac{\partial a}{\partial u}\right)\right\}\right]$$

$$+k\left[\left(\xi\frac{\partial a}{\partial u}\right)\frac{\partial r}{\partial v}-\left(\xi\frac{\partial a}{\partial v}\right)\frac{\partial r}{\partial u}\right],$$

$$C=\sin\frac{r}{k}\cos\frac{r}{k}\left[\left(\xi\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial\xi}{\partial v}\right)-\left(\xi\frac{\partial a}{\partial v}\right)\left(\eta\frac{\partial\xi}{\partial u}\right)\right.$$

$$\left.+k^2\left\{\frac{\partial}{\partial v}\left(\eta\frac{\partial a}{\partial u}\right)-\frac{\partial}{\partial u}\left(\eta\frac{\partial a}{\partial v}\right)\right\}\right]$$

$$+k\left[\left(\eta\frac{\partial a}{\partial v}\right)\frac{\partial r}{\partial u}-\left(\eta\frac{\partial a}{\partial u}\right)\frac{\partial r}{\partial v}\right].$$

The equations of condition that the congruence be cyclical are consequently

$$A=0,\quad B=0,\quad C=0.$$

For a focal congruence, these conditions will be (taking u and v as focal parameters)

$$1+\lambda\lambda'\sin^2\frac{r}{k}\cos^2\frac{r}{k}=0, \tag{32}$$

[1] Kashiwagi, Loc. cit.; V. Sbrana, Rendiconti del circolo Math. **21** (1906).

or

$$\left(\xi\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial a}{\partial v}\right)-\left(\eta\frac{\partial a}{\partial u}\right)\left(\xi\frac{\partial a}{\partial v}\right)=0,$$

$$\left(\xi\frac{\partial a}{\partial u}\right)\frac{\partial r}{\partial v}-\left(\xi\frac{\partial a}{\partial v}\right)\frac{\partial r}{\partial u}=0, \tag{33}$$

$$\left(\eta\frac{\partial a}{\partial u}\right)\frac{\partial r}{\partial v}-\left(\eta\frac{\partial a}{\partial v}\right)\frac{\partial r}{\partial u}=0.$$

But from the first condition (33), we have

$$\boldsymbol{EG-F^2=E'G'-F'^2=-(eg-ff')}$$

i.e. the two focal spheres are mutually orthogonal.

When the second conditions are satisfied, the two equations (17) must be dependent. So all the circles of the congruence must pass through a point-pair.

Hence we have the theorem.

Theorem. A necessary and sufficient condition that a focal congruence should form a cyclical system is that it should consist of circles through a point-pair or else that the focal spheres through every circle should be mutually orthogonal.

PSEUDO-NORMAL CONGRUENCE.

9. If the congruence of space-axes of a focal congruence be normal, we must have[1]

$$f=f'.$$

And from (27), we get (taking as u and v the focal parameters)

$$\lambda=\lambda', \tag{34}$$

and we have

$$dy_i=px_i+qy_i+rdz_i, \quad (i=0,\ 1,\ 2,\ 3,\ 4).$$

This gives the conclusion that any two adjacent circles of the congruence will be cospherical i.e. all the circles lie on a sphere, or the congruence will consist of circles through a point-pair.

[1] Fibbi, loc. cit.; Coolidge, loc. cit.

Theorem. A necessary and sufficient condition that the congruence of space axes of the focal congruence should be normal is that they should consist of circles through a point-pair on a sphere.

In this case, we have from (25)

$$\left(a\frac{\partial b}{\partial u}\right)\frac{\partial r}{\partial v}=\left(a\frac{\partial b}{\partial v}\right)\frac{\partial r}{\partial u}.$$

Hence

$$f=f'.$$

10. *Definition. Two points of a sphere are said to be inverse in a circle thereof, if all the circles of the sphere through them are orthogonal to that circle.*

If the points of contact of the main sphere[1] with the two nappes of its envelope are mutually inverse in the circle, we should have

$$\begin{Vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{Vmatrix} \cdot \begin{Vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ \frac{\partial x_0}{\partial u} & \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} & \frac{\partial x_4}{\partial u} \end{Vmatrix} = 0,$$

$$\begin{Vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{Vmatrix} \cdot \begin{Vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ \frac{\partial x_0}{\partial v} & \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} & \frac{\partial x_4}{\partial v} \end{Vmatrix} = 0.$$

Or

$$\left(y\frac{\partial x}{\partial u}\right)=\left(y\frac{\partial x}{\partial v}\right)=0.$$

i.e.

$$\left(b\frac{\partial a}{\partial u}\right)=\left(b\frac{\partial a}{\partial v}\right)=0.$$

That is, the plane of the circle is the tangent to the surface described by the center of the circle of congruence. And it is evident that in such a case the space axes of congruence are normal and

$$f=f'.$$

11. *Definition. We shall call a congruence to be a pseudo-normal when through each circle of the congruence it is possible to pass a sphere whose points of contact with its envelope are mutually inverse in that circle.*

[1] Nishiuchi and Kashiwagi, loc. cit.

Suppose that such a sphere through our circle have the coordinates

$$\bar{x}_i = x_i \cos\phi + y_i \sin\phi, \qquad (i=0,\ 1,\ 2,\ 3,\ 4)$$

We must have

$$\begin{Vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{Vmatrix} \cdot \begin{Vmatrix} \bar{x}_0 & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 \\ \frac{\partial \bar{x}_0}{\partial u} & \frac{\partial \bar{x}_1}{\partial u} & \cdots\cdots & & \frac{\partial \bar{x}_4}{\partial u} \end{Vmatrix} = 0,$$

$$\begin{Vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{Vmatrix} \cdot \begin{Vmatrix} \bar{x}_0 & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 \\ \frac{\partial \bar{x}_0}{\partial v} & \frac{\partial \bar{x}_1}{\partial v} & \cdots\cdots & & \frac{\partial \bar{x}_4}{\partial v} \end{Vmatrix} = 0,$$

or

$$\left(y\frac{\partial \bar{x}}{\partial u}\right)\cos\phi - \left(x\frac{\partial \bar{x}}{\partial u}\right)\sin\phi = 0,$$

$$\left(y\frac{\partial \bar{x}}{\partial v}\right)\cos\phi - \left(x\frac{\partial \bar{x}}{\partial v}\right)\sin\phi = 0,$$

and we get

$$\left(y\frac{\partial x}{\partial u}\right) = -k^2\frac{\partial \phi}{\partial u},$$

(35)

$$\left(y\frac{\partial x}{\partial v}\right) = -k^2\frac{\partial \phi}{\partial v},$$

i.e.

(36) $$f = f'.$$

The diffential equations (35) are equally well satisfied if we replace ϕ by $\phi + c$, where c is any finite constant.

Hence we have the following:

In pseudo-normal congruence an infinite number of spheres will pass through each circle, whose points of contact with its envelope are mutually inverse in that circle.

12. Suppose that we have a congruence of circles which is both focal and pseudo-normal. We find from (28) and (36) that

$$(\lambda - \lambda')f = 0.$$

That is

$$\lambda = \lambda',$$

or

$$f = 0.$$

We have

$$f = f' = F = F' = 0, \tag{37}$$

and the equation for the limiting spheres will be

$$\left(\lambda \sin\frac{r}{k}\cos\frac{r}{k}\tan\phi + 1\right)\left(\tan\phi - \lambda\sin\frac{r}{k}\cos\frac{r}{k}\right)$$

$$\times\left(\lambda' \sin\frac{r}{k}\cos\frac{r}{k}\tan\phi + 1\right)\left(\tan\phi - \lambda'\sin\frac{r}{k}\cos\frac{r}{k}\right) = 0.$$

Hence the focal spheres will coincide with the pair of limiting spheres.

Theorem. A necessary and sufficient condition that a focal congruence should be pseudo-normal is that the space axes of our circles of congruence should be normal or that the focal spheres should coincide with a pair of limiting spheres.

13. Let (α) and (β) be the coordinates of the centers of the foci of our circle, then

$$\begin{aligned} \alpha_i &= a_i \sec\frac{r}{k} + ib_i\tan\frac{r}{k}, \\ \beta_i &= a_i\sec\frac{r}{k} - ib_i\tan\frac{r}{k}, \\ (\alpha\alpha) = (\beta\beta) &= k^2, \quad (i=0,\ 1,\ 2,\ 3) \end{aligned} \tag{38}$$

and

$$\begin{aligned} \left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right) &= \frac{1}{k}\tan^2\frac{r}{k}\left[(F - F') + i(f + f')\right], \\ \left(\frac{\partial\beta}{\partial u}\frac{\partial\beta}{\partial v}\right) &= \frac{1}{k}\tan^2\frac{r}{k}\left[(F - F') - i(f + f')\right]. \end{aligned} \tag{39}$$

Hence if a congruence be both focal and pseudo-normal we have (taking the focal parameter as u and v)

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right) = \left(\frac{\partial\beta}{\partial u}\frac{\partial\beta}{\partial v}\right) = 0.$$

And moreover, as

$$a_i = \frac{\alpha_i + \beta_i}{2\sec\dfrac{r}{k}}, \qquad b_i = \frac{\alpha_i - \beta_i}{2i\tan\dfrac{r}{k}} \qquad (i=0,\ 1,\ 2,\ 3) \tag{40}$$

we have from (24)

$$A\frac{\partial\beta_i}{\partial u} = B(\alpha_i - \beta_i) + C\frac{\partial\alpha_i}{\partial u},$$

(41)

$$A'\frac{\partial\beta_i}{\partial v} = B'(\alpha_i - \beta_i) + C'\frac{\partial\alpha_i}{\partial v}$$

and

$$p\frac{\partial^2\alpha_i}{\partial u\partial v} = q(\alpha_i - \beta_i) + r\frac{\partial\alpha_i}{\partial u} + s\frac{\partial\alpha_i}{\partial v},$$

$$p'\frac{\partial^2\beta_i}{\partial u\partial v} = q'(\alpha_i - \beta_i) + r'\frac{\partial\beta_i}{\partial u} + s'\frac{\partial\beta_i}{\partial v}.$$

But since

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right) = -\left(\alpha\frac{\partial^2\alpha}{\partial u\partial v}\right) = 0,$$

$$\left(\frac{\partial\beta}{\partial u}\frac{\partial\beta}{\partial v}\right) = -\left(\beta\frac{\partial^2\beta}{\partial u\partial v}\right) = 0.$$

we get

$$q = q' = 0.$$

and

$$p\frac{\partial^2\alpha_i}{\partial u\partial v} = r\frac{\partial\alpha_i}{\partial u} + s\frac{\partial\alpha_i}{\partial v},$$

(42)

$$p'\frac{\partial^2\beta_i}{\partial u\partial v} = r'\frac{\partial\beta_i}{\partial u} + s'\frac{\partial\beta_i}{\partial v}.$$

Or

$$\begin{vmatrix} \alpha & \frac{\partial\alpha}{\partial u} & \frac{\partial\alpha}{\partial v} & \frac{\partial^2\alpha}{\partial u\partial v} \end{vmatrix} = 0,$$

$$\begin{vmatrix} \beta & \frac{\partial\beta}{\partial u} & \frac{\partial\beta}{\partial v} & \frac{\partial^2\beta}{\partial u\partial v} \end{vmatrix} = 0.$$

And we have the theorem.

Theorem. A condition that a focal congruence should be pseudo-normal is that the focal parameters should give the lines of curvature of the surfaces described by the centers of foci.

CONFORMAL CONGRUENCE.

14. We shall now abandon the general congruence and assume that

$$(43) \qquad e : \frac{f+f'}{2} : g = E-E' : F-F' : G-G'.$$

Let (α) and (β) be the coordinates of centers of foci, then from the relations (38)

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial u}\right) = \frac{r}{k^2}\tan^2\frac{r}{k}\left[(E-E')+2ie\right],$$

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right) = \frac{1}{k^2}\tan^2\frac{r}{k}\left[(E-F')+i(f+f')\right],$$

$$\left(\frac{\partial\alpha}{\partial v}\frac{\partial\alpha}{\partial v}\right) = \frac{1}{k^2}\tan^2\frac{r}{k}\left[(G-G')+2ig\right],$$

$$\left(\frac{\partial_1\beta}{\partial u}\frac{\partial\beta}{\partial u}\right) = \frac{1}{k^2}\tan^2\frac{r}{k}\left[(E-E')-2ie\right],$$

$$\left(\frac{\partial_1\beta}{\partial u}\frac{\partial\beta}{\partial v}\right) = \frac{1}{k^2}\tan^2\frac{r}{k}\left[(F-F')-i(f+f')\right],$$

$$\left(\frac{\partial_1\beta}{\partial v}\frac{\partial\beta}{\partial v}\right) = \frac{1}{k^2}\tan^2\frac{r}{k}\left[(G-G')-2ig\right].$$

And in this care, we have

$$\frac{\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial u}\right)}{\left(\frac{\partial_1\beta}{\partial u}\frac{\partial\beta}{\partial u}\right)} = \frac{\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right)}{\left(\frac{\partial_1\beta}{\partial u}\frac{\partial\beta}{\partial v}\right)} = \frac{\left(\frac{\partial\alpha}{\partial v}\frac{\partial\alpha}{\partial v}\right)}{\left(\frac{\partial_1\beta}{\partial v}\frac{\partial\beta}{\partial v}\right)},$$

$$(d\alpha\, d\alpha) = \rho(d_1\beta\, d_1\beta).$$

If these various expressions do not all vanish identically, the two surfaces of centers of foci are conformally related. If they do vanish identically, (α) and (β) trace two isotropic curves, given respectively by the parameters u and v.

Suppose, conversely, that (α) and (β) are corresponding points on two conformally related surfaces,

$$(d\alpha d\alpha) \equiv \rho(d_1\beta d_1\beta),$$

$$(\alpha_1\beta) = k^2\left(1+2\tan^2\frac{r}{k}\right)$$

and we have the relation (43) easily.

In these equations above, if we take

$$\rho = 1$$

i.e. if two surfaces (α) and (β) applicable, we get

$$e = f + f' = g = 0.$$

And the equation (10) reduces to

$$[(E - E')du^2 + 2(F - F')dudv + (G - G')dv^2]\sin\phi\cos\phi = 0,$$

a circle orthogonal and cospherical with our given circles, and one of its next neighbours must lie either (x) or (y).

We next write

$$x_i' = x_i\cos\phi + y_i\sin\varphi,$$
$$x_i'' = x_i\cos\phi - y_i\sin\varphi,$$
$$(i = 0,\ 1,\ 2,\ 3,\ 4;\ \varphi = \text{const.})$$

we have

$$(dx'\,dx') \equiv (dx''\,dx'').$$

Conversely, if this equation hold, and if φ be constant,

$$(dxdy) \equiv 0.$$

We shall say that two congruences of spheres are *conformally related* if the angle of two adjacent spheres of the one be equal to the corresponding angle in the other.

Theorem. If the foci of the circles of a congruence be corresponding points on two applicably related surfaces, or trace two isotropie curves, then, through each circle of the congruence we may, in an infinite number of ways, pass two spheres which shall make a constant angle with one another, and describe a conformally related congruence.

15. Let us no now make the additional assumption that our congruence is focal.

If a congruence be both conformal and pseudo-normal

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right) = \left(\frac{\partial\beta}{\partial u}\frac{\partial\beta}{\partial v}\right) = 0,$$

$$\frac{\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial u}\right)}{\left(\frac{\partial\beta}{\partial u}\frac{\partial\beta}{\partial u}\right)} = \frac{\left(\frac{\partial\alpha}{\partial v}\frac{\partial\alpha}{\partial v}\right)}{\left(\frac{\partial\beta}{\partial v}\frac{\partial\beta}{\partial v}\right)}.$$

This gives

$$(\lambda-\lambda')\sin\frac{r}{k}\cos\frac{r}{k}(1+\lambda\lambda'\sin^2\frac{r}{k})=0.$$

Hence we have the theorem.

Theorem. If a congrueuce be both conformal and pseudo-normal it is normal or the congruence of space axes of our circles is normal.

In conclusion the author wishes to express his sincere thanks to Prof. T. Nishiuchi for his kind guidance and remarks.

On Transcendental Integral and Transcendental Algebraic Functions and Algebraic Addition Theorems, I.

By

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Our main problem is the study of the analytic functions of many independent variables, which have algebraic addition theorems. For that purpose we shall first discuss some properties of transcendental integral and transcendental algebraic functions, which are of fundamental importance for our later investigations.

Chapter I.

TRANSCENDENTAL INTEGRAL FUNCTIONS OF TRANSFINITE ORDERS.

INTRODUCTION.

Suppose that $F(z)$ be a transcendental integral function and $M(r)$ be the maximum value of its modulus for $|z|=r$. Then there exist two finite numbers α, β such that $e^{r^\alpha} < M(r) < e^{r^\beta}$, or $\dfrac{\log\log M(r)}{\log r}$ is a number in the interval (α, β). In the case that β is limited for all values of r, we define, following Prof. Borel,[1] that when $\lim\limits_{r=\infty} \dfrac{\log\log M(r)}{\log r}$ is determinate, $F(z)$ is said to be *a regularly increasing function* (la

[1] Borel, *Leçons sur les fonctions entières*, p. 107.

fonction à croissance régulière), and in the other case, *an irregularly increasing function* (la fonction à croissance irrégulière). The upper limit ρ of $\frac{\log\log M(r)}{\log r}$ for limit $r=\infty$ is called the *order* of $F(z)$. In this case the order of infinitude of the moduli of the zero points of $F(z)$ is determinate and is generally equal to the inverse of its order; and conversely.

We may exdend this conception to the case where ρ is transfinite.

Definition. Let $\log_2=\log\log$, ..., $\log_p=\log\log.....\log$; p: any positive integer. When $\overline{\lim_{r=\infty}}\frac{\log_{p-1}M(r)}{\log r}=\infty$, and $\overline{\lim_{r=\infty}}\frac{\log_p M(r)}{\log r}=\rho$ is finite, we say that the *order* of $F(z)$ is $\omega^{p-2}\cdot\rho$ (e.g. the orders of e^{e^z} and $e^{e^{z^2}}$ are ω and $\omega\cdot 2$ respectively). We define also that $F(z)$ is *regularly* or *irregularly increasing* according as $\lim_{r=\infty}\frac{\log_p M(r)}{\log r}$ is or is not determinate.

In the following, we deal with only those transcendental integral functions whose orders are less than $\Omega=\omega^\omega$.

1. Let $F(z)$ be an entire function whose zero points are the origin (of multiplicity λ) and $z=a_1, a_2, ..., a_n, ...$. Suppose that $r_1\leqq r_2\leqq.....\leqq r_n\leqq.....$, where $r_n=|a_n|$, and consider the series $\sum_{n=1}^{\infty}\frac{1}{r^\alpha}$ where α is a certain positive number. If $\sum_{n=1}^{\infty}\frac{1}{r_n^\alpha}$ be divergent for any positive value of α (however great), then we consider $\sum_{n=1}^{\infty}\frac{1}{e^{r_n^\alpha}}$, and so on.

Definition. Let e_p^x be an inverse function of $\log_p x$. If a positive integer $p'(\geqq 2)$ and a positive value (zero being included) ρ' be such that, for any prescribed positive value ε (however small), $\sum_{n=1}^{\infty}\frac{1}{e_{p'-2}^{r_n^{\rho'-\varepsilon}}}$ [1] is divergent while $\sum_{n=1}^{\infty}\frac{1}{e_{p'-2}^{r_n^{\rho'+\varepsilon}}}$ is convergent, then the *exponent of con-*

[1] For $\rho'=0$, we take $\sum_{n=1}^{\infty}\frac{1}{e_{p'-3}^{r_n^G}}$ instead of $\sum_{n=1}^{\infty}\frac{1}{e_{p'-2}^{r_n^{\rho'-\varepsilon}}}$ where G is any positive value.

vergency of the moduli of the zero points of $F(z)$ is said to be $\omega^{p'-2}\rho'$. This definition is an extension of that for $p'=2$.[1]

Lemma. If a series $\sum_{n=1}^{\infty} u_n$ *whose terms are positive and decreasing be convergent, then* $\lim_{n=\infty} nu_n=0$.[2]

Theorem 1. Let the zero points of a transcendental integral function $F(z)$ *be* $a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$ and $0<r_n\leqq r_{n+1}$ $(n=1, 2, \ldots\ldots)$ *where* $r_n=|a_n|$. *If the exponent of convergency of* $r_1, r_2, \ldots\ldots r_n, \ldots\ldots$ *be* $\omega^{p'-2}\rho'$, *the upper limit of* $\dfrac{\log_{p'-1} n}{\log r_n}$ *when* n *is infinite will be equal to* ρ'.

If $\overline{\lim_{n=\infty}}\dfrac{\log n}{\log r_n}=\infty$, we consider $\overline{\lim_{n=\infty}}\dfrac{\log_2 n}{\log r_n}$ and so on. Suppose that $\overline{\lim_{n=\infty}}\dfrac{\log_{p_1-1} n}{\log r_n}=\rho_1$ is finite while $\overline{\lim_{n=\infty}}\dfrac{\log_{p_1-2} n}{\log r_n}=\infty$. Then for any prescribed positive value ε, there is a corresponding positive number N such that

$$\frac{\log_{p_1-1} n}{\log r_n}\leqq \rho_1+\frac{\varepsilon}{2} \qquad \text{for} \quad . \quad n\geqq N.$$

Accordingly there is a positive value δ such that

$$\frac{\log_{p_1-1}(n^{1+\delta})}{\log r_n}\leqq \rho_1+\varepsilon \qquad \text{for} \qquad n\geqq N,$$

or

$$n^{1+\delta}\leqq e^{r_n^{\rho_1+\varepsilon}}_{p_1-2} \qquad \text{for} \qquad n\geqq N,$$

so that

$$\sum_{n=N}^{\infty}\frac{1}{n^{1+\delta}}\geqq\sum_{n=N}^{\infty}\frac{1}{e^{r_n^{\rho_1+\varepsilon}}_{p_1-2}}.$$

But $\sum_{n=1}^{\infty}\dfrac{1}{n^{1+\delta}}$ is convergent and accordingly so also for $\sum_{n=1}^{\infty}\dfrac{1}{e^{r_n^{\rho_1+\varepsilon}}_{p_1-2}}$.

Therefore, we have

$$\omega^{p_1-2}(\rho_1+\varepsilon)\geqq\omega^{p'-2}\rho',$$

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[1] Borel, *loc. cit.*, p. 18.

[2] Borel, *loc. cit.*, p. 17.

which is valid for any positive value of ε, so that

$$\omega^{p_1-2}\cdot\rho_1 \geqq \omega^{p'-2}\cdot\rho'. \tag{1}$$

Next, we consider the convergent series $\sum_{n=1}^{\infty}\frac{1}{r_n^{\rho'+\varepsilon}\,e^{p'-2}}$. By the foregoing lemma, we have $\lim_{n=\infty}\frac{n}{r_n^{\rho'+\varepsilon}\,e^{p'-2}}=0$, from which it follows that

$$n<e^{r_n^{\rho'+\varepsilon}}_{p'-2} \qquad \text{for} \qquad n\geqq N(\varepsilon),$$

where $N(\varepsilon)$ is a certain positive value which depends on ε. Therefore

$$\overline{\lim_{n=\infty}}\frac{\log_{p'-1}n}{\log r_n}<\rho'+\varepsilon. \tag{2}$$

We now suppose that $p_1>p'$. As

$$\overline{\lim_{n=\infty}}\frac{\log_{p_1-2}n}{\log r_n}=\overline{\lim_{n=\infty}}\left(\frac{\log_{p_1-2}n}{\log_{p_1-3}n}\cdot\frac{\log_{p_1-3}n}{\log_{p_1-4}n}\cdots\frac{\log_{p'}n}{\log_{p'-1}n}\cdot\frac{\log_{p'-1}n}{\log r_n}\right)$$

and

$$\overline{\lim_{n=\infty}}\frac{\log_{p_1-2}n}{\log_{p_1-3}n}=0,\ \ldots,\ \overline{\lim_{n=\infty}}\frac{\log_{p'}n}{\log_{p'-1}n}=0,\ \overline{\lim_{n=\infty}}\frac{\log_{p'-1}n}{\log r_n}<\rho'+\varepsilon \text{ by (2)},$$

we have

$$\overline{\lim_{n=\infty}}\frac{\log_{p_1-2}n}{\log r_n}<\rho'+\varepsilon,$$

which is contrary to the assumption that

$$\overline{\lim_{n=\infty}}\frac{\log_{p_1-2}n}{\log r_n}=\infty.$$

Hence

$$p_1=p', \tag{3}$$

so that we have by (1)

$$\rho_1\geqq\rho'. \tag{1}'$$

As, by (2) and (3),

$$\rho_1=\overline{\lim_{n=\infty}}\frac{\log_{p_1-1}n}{\log r_n}=\overline{\lim_{n=\infty}}\frac{\log_{p'-1}n}{\log r_n}<\rho'+\varepsilon,$$

we have $\rho_1<\rho'+\varepsilon$, which holds for any positive value of ε; so that

$$\rho_1\leqq\rho'. \tag{4}$$

By (1)′ and (4), we have $\rho_1=\rho$, proving the proposition.

Similarly, supposing that $\underline{\lim_{n=\infty}}\dfrac{\log_{p_2-2}n}{\log r_n}=\infty$, while $\underline{\lim_{n=\infty}}\dfrac{\log_{p_2-1}n}{\log r_n}=\rho_2$, we have $\omega^{p_2-2}\cdot\rho_2\leqq\omega^{p'-2}\cdot\rho'$.

Definition. Let the zero points of a transcendental integral function $F(z)$ be $a_1, a_2, \ldots, a_n, \ldots$, and $0<r_n\leqq r_{n+1}\ (n=1, 2, \ldots\ldots)$ where $r_n=|a_n|$. Supposing that the exponent of convergency of $r_1, r_2, \ldots, r_n, \ldots$ be $\omega^{p'-2}\cdot\rho'$, if $\overline{\lim_{n=\infty}}\dfrac{\log_{p'-1}n}{\log r_n}=\underline{\lim_{n=\infty}}\dfrac{\log_{p'-1}n}{\log r_n}=\rho'$, we say that *the order of infinitude of* $r_n\ (n=1, 2, \ldots\ldots)$ *is determinate* and is equal to $\dfrac{1}{\rho'}\cdot\dfrac{1}{\omega^{p'-2}}$, and in the other case, *indeterminate.*

2. Let the greatest integral value of n which satisfies at least one of $r_n\leqq 1$ and $n\leqq e^1_{p'-2}$ be n_1, and we determine the integers $p_n's$ as follows:

$$\begin{cases} p_n=0, & \text{for } n\leqq n_1 \\ p_n<\dfrac{\log n\log_2 n\ldots\ldots\log_{p'-1}n}{\log r_n}\leqq p_n+1, & \text{for } n>n_1. \end{cases} \tag{5}$$

Let

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

i) Supposing that $p'=2$, we have $\overline{\lim_{n=\infty}}\dfrac{\log n}{\log r_n}=\rho'$. If ρ' be not an integer, we take $p_n<\rho'<p_n+1$ for $n=1, 2, \ldots\ldots$. Then $p_n's$ thus determined will satisfy the lower relation in (5) for sufficiently great values of n, provided that there is no integer which is less than $\overline{\lim_{n=\infty}}\dfrac{\log n}{\log r_n}=\rho'$ and is not less than $\underline{\lim_{n=\infty}}\dfrac{\log n}{\log r_n}$. If ρ' be an integer, we take $p_n=\rho'$ or $\rho'-1$, $(n=1, 2, \ldots\ldots)$, according as $\sum_{n=1}^{\infty}\dfrac{1}{r_n^{\rho'}}$ is

divergent or convergent. Thus, $\sum_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$ is convergent uniformly and unconditionarily for all finite values of z,[1] so that

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

is an entire function whose zero points are the origin (of multiplicity λ) and $z=a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$.

ii) Next, we suppose that $p'\geqq 3$. For any arbitrarily assigned value of z, whose absolute value is r, determine an integer n_2 such that

$$r_{n_2}^{1-\frac{1}{\log_{p'-1}n_2}}\leqq r<r_{n_2+1}^{1-\frac{1}{\log_{p'-1}(n_2+1)}}. \tag{6}$$

Let the greater of n_1 and n_2 be N, and put

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

$$\equiv z^{\lambda}\prod_{n=1}^{N}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}\prod_{N+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

in which $z^{\lambda}\prod_{n=1}^{N}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^3+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{p_n}\right)^{p_n}}$ is an entire function.

As $r<r_{n_2+1}^{1-\frac{1}{\log_{p'-1}(n_2+1)}}$ by (6), there is a positive value ε which satisfies

$$r=r_{n_2+1}^{1-\frac{1+\varepsilon}{\log_{p'-1}(n_2+1)}}. \tag{7}$$

Moreover as $r_1\leqq r_2\leqq\cdots\cdots\leqq r_n\leqq\cdots\cdots$, we have

$$r_n^{1-\frac{1+\varepsilon}{\log_{p'-1}n}}<r_{n+1}^{1-\frac{1+\varepsilon}{\log_{p'-1}(n+1)}} \quad \text{for} \quad n>N, \tag{8}$$

and accordingly by (7) and (8)

$$\log r_n-\log r\geqq(1+\varepsilon)\frac{\log r_n}{\log_{p'-1}n} \quad \text{for} \quad n>N. \tag{9}$$

[1] Forsyth, *Theory of Functions*, 3rd edition, p. 94.

We have however by (5)

$$p_n+1 \geqq \frac{\log n \log_2 n \ldots\ldots \log_{p'-1} n}{\log r_n} \quad \text{for } n>N\ (\geqq n_1),$$

so that by (9)

$$(p_n+1)(\log r_n - \log r) \geqq \frac{\log n \log_2 n \ldots\ldots \log_{p'-1} n}{\log r_n}(1+\varepsilon)\frac{\log r_n}{\log_{p'-1} n}$$

$$\geqq (1+\varepsilon)\log n \qquad \text{for } n>N,$$

or

$$\left(\frac{r_n}{r}\right)^{p_n+1} \geqq n^{1+\varepsilon} \qquad \text{for } n>N.$$

Accordingly

$$\sum_{n=N+1}^{\infty}\left(\frac{r}{r_n}\right)^{p_n+1} \leqq \sum_{n=N+1}^{\infty}\frac{1}{n^{1+\varepsilon}} < \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} \qquad \text{(convergent)},$$

and we may put

$$(10) \qquad \sum_{n=N+1}^{\infty}\left(\frac{r}{r_n}\right)^{p_n+1} = A.$$

As $\overline{\lim_{n=\infty}} \dfrac{\log_{p'-1} n}{\log r_n} = \rho'$, there is a finite positive value B such that $\dfrac{\log_{p'-1} n}{\log r_n} < B$ for $n>N$, and we have by (6)

$$r < r_n^{1-\frac{1}{\log_{p'-1} n}} = r_n e^{-\frac{\log r_n}{\log_{p'-1} n}} < r_n e^{-\frac{1}{B}} \quad \text{for } n>N, \text{ so that}$$

$$(11) \qquad 1-\frac{r}{r_n} > 1-e^{-\frac{1}{B}} \qquad \text{for } n>N.$$

Now

$$\left|\prod_{n=N+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}\right|$$

$$\leqq \prod_{n=N+1}^{\infty} e^{\frac{1}{p_n+1}\left(\frac{r}{r_n}\right)^{p_n+1}+\frac{1}{p_n+2}\left(\frac{r}{r_n}\right)^{p_n+2}+\ldots\ldots}$$

$$< \prod_{n=N+1}^{\infty} e^{\left(\frac{r}{r_n}\right)^{p_n+1}+\left(\frac{r}{r_n}\right)^{p_n+2}+\ldots\ldots\ldots}$$

$$= \prod_{n=N+1}^{\infty} e^{\frac{\left(\frac{r}{r_n}\right)^{p_n+1}}{1-\frac{r}{r_n}}}$$

$$< e^{\frac{A}{1-e^{-B}}}, \qquad \text{by (10) and (11).}$$

Consequently $\prod_{n+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$ is convergent uniformly and unconditionally for all finite values of z, so that

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

is an entire function whose zero points are the origin (of multiplicity λ) and $z=a_1, a_2, \ldots\ldots, a_n \ldots\ldots$.

In all cases $f(z)$ is an entire function which has the same zero points of the same orders as $F(z)$. Hence $\frac{F(z)}{f(z)}$ is an entire function which has no zero points, and we may put $\frac{F(z)}{f(z)}\equiv e^{Q(z)}$, where $Q(z)$ is an entire function. Thus $F(z)$ *may be written in the form*

$$F(z)\equiv z^{\lambda}e^{Q(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{2}\left(\frac{z}{p_n}\right)^{p_n}},$$

where $Q(z)$ is an entire function and p_n's are integers determined as in (5).

3. *Theorem* 2. *Let*

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

be an entire function whose exponent of convergency of the moduli of the zero points is $w^{p'-2}\rho'$, where p_n's are determined as to satisfy (5). *Then for any prescribed positive value* ε, *there corresponds a positive value R such that*

$$|f(z)|\leqq e^{r^{\rho'+\varepsilon}} \qquad \text{for all} \quad |z|=r\geqq R,$$

that is, the order of $f(z)$ can not exceed the exponent of convergency.

The proof of this theorem for $p'=2$ is given in Borel, *loc. cit.*, p. 61, and the following proof is for $p'\geqq 3$. Determine n_1 and n_2 as

in §2. Then for sufficiently great values of r, we have $n_2 \geqq n_1$, and

$$f(z) \equiv z^{\lambda} \prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right) \prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

$$\times \prod_{n_2+1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

$z^{\lambda} \prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)$ being a polynominal of the $(n_1+\lambda)^{th}$ degree, we have

$$\left| z^{\lambda} \prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right) \right| \leqq e^{r^{\rho'}}_{p'-1}$$

for all values of r greater than R which is a suitably chosen positive value. Secondly, we consider

$$\prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

$$\left| \prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}} \right|$$

$$\leqq \prod_{n_1+1}^{n_2}\left(1+\frac{r}{r_n}\right) e^{\frac{r}{r_n}+\frac{1}{2}\left(\frac{r}{r_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{r}{r_n}\right)^{p_n}}$$

$$< \prod_{n_1+1}^{n_2} e^{2\frac{r}{r_n}+\frac{1}{2}\left(\frac{r}{r_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{r}{r_n}\right)^{p_n}}$$

$$< \prod_{n_1+1}^{n_2} e^{(p_n+1)r^{p_n}}$$

$$< e^{n_2(p_k+1)r^{p_k}},$$

where p_k is the greatest of p_{n_1+1}, p_{n_1+2}, ..., p_{n_2}. We have, however, by (5)

$$p_n < \frac{\log n \log_2 n \cdots\cdots \log_{p'-1} n}{\log r_n} \qquad \text{for} \quad n > n_1,$$

so that $p_n < \log n \log_2 n \cdots\cdots \log_{p'-1} n = (\log n)^{1+\varepsilon'}$ where ε' is a positive value which can be made as small as we please by taking n sufficiently great. Accordingly

$$\left|\prod_{n_1+1}^{n_2}\right| < e^{n_2\left\{(\log k)^{1+\varepsilon'}+1\right\}r^{1+\varepsilon'}} \leqq e^{n_2\left\{(\log n_2)^{1+\varepsilon'}+1\right\}r^{1+\varepsilon'}}.$$

As $\overline{\lim_{n=\infty}}\dfrac{\log_{p'-1}n}{\log r_n}=\rho'$ and $r^{1-\frac{1}{\log_{p'-1}n}}\leqq r$ by (6), we have $n_2 \leqq e^{r^{\rho'+\varepsilon''}}_{p'-2}$ where ε'' is a certain positive value which can be made as small as we please by taking n_2 sufficiently great; so that

$$\left|\prod_{n_1+1}^{n_2}\right| \leqq e^{e^{r^{\rho'+\varepsilon''}}_{p'-2}\cdot\left\{\left(e^{r^{\rho'+\varepsilon''}}_{p'-3}\right)^{1+\varepsilon'}+1\right\}e^{\left(e^{r^{\rho'+\varepsilon''}}_{p'-3}\right)^{1+\varepsilon'}\log r}} \leqq e^{r^{\rho'+\varepsilon_1}}_{p'-1}$$

where ε_1 is a certain positive value of the same property as ε''. As n_2 increases with r, we have

$$\left|\prod_{n_1+1}^{n_2}\right| \leqq e^{r^{\rho'+\varepsilon_1}}_{p'-1}$$

for $r \geqq R_2$, where R_2 is a certain positive value which corresponds to ε_1.

Lastly, for $\prod_{n_2+1}^{\infty}\left(1-\dfrac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$, we have only to repeat the process carried out in §2 and the same result must follow

$$\left|\prod_{n_2+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}\right| < e^{1-e^{-\frac{1}{B}}},$$

or

$$\left|\prod_{n_2+1}^{\infty}\right| \leqq e^{r^{\rho'}}_{p'-1}$$

for $r \geqq R_3$, where R_3 is a suitably chosen positive value. We have accordingly

$$\left|f(z)\right| \equiv \left|z^{\lambda}\prod_{n=1}^{n_1}\right|\cdot\left|\prod_{n_1+1}^{n_2}\right|\cdot\left|\prod_{n_2+1}^{\infty}\right|$$

$$\leqq e^{r^{\rho'}}_{p'-1}\cdot e^{r^{\rho'+\varepsilon_1}}_{p'-1}\cdot e^{r^{\rho'}}_{p'-1}$$

for $r>R_1$, R_2, R_3, or

$$\left|f(z)\right| \leqq e^{r^{\rho'+\varepsilon}_{p'-1}}$$

for all value of $r>R$, where R is a certain positive value corresponding to ε.

4. *Lemma. Let the zero points of a transcendental integral function* $\varphi(z)$ *be* $z=a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$ *and* $0<r_n \leqq r_{n+1}$ $(n=1, 2, \ldots\ldots)$ *where* $r_n=|a_n|$. *For an arbitrarily assigned value of* $|z|=r$, *find* r_n *as to satisfy*

$$sr_n \leqq r \leqq sr_{n+1},$$

where s *is a certain integer greater than* 2. *Then*

$$n\log(s-1)<\log M_1(r),$$

where $M_1(r)$ *is the maximum value of* $|\varphi(z)|$ *for* $|z|=r$.

The proof of this lemma is given in Borel, *loc. cit.*, p. 73.

Theorem 3. *Extension of Hadamard's first theorem*[1] *to entire functions of the transfinite orders.*

Let $F(z)$ *be an entire function of the transfinite order* $\omega^{p-2}\rho$ *and the exponent of convergency of the moduli of the zero points of* $F(z)$ *be* $\omega^{p'-2}\rho'$. *Then* $\omega^{p'-2}\rho' \leqq \omega^{p-2}\rho$.

Suppose that the zero points of $F(z)$, which are different from the origin, be $z=a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$, and that $0<r_n \leqq r_{n+1}(n=1, 2, \ldots\ldots)$ where $r_n=|a_n|$. Then the exponent convergency of $r_1, r_2, \ldots\ldots$is $\omega^{p'-2}\rho'$. As $F(z)$ is of the order $\omega^{p-2}\rho$, for any prescribed positive value ε, there corresponds a positive value R such that

$$\left|F(z)\right| \leqq e^{r^{\rho+\varepsilon}_{p-1}} \qquad \text{for} \qquad |z|=r \geqq R.$$

We have, however, by the lemma

$$\begin{aligned} n\log(s-1) &< \log M(r) \\ &\leqq \log e^{r^{\rho+\varepsilon}_{p-1}} \\ &= e^{r^{\rho+\varepsilon}_{p-2}} \end{aligned}$$

for $r \geqq R$, where $sr_n \leqq r \leqq sr_{n+1}$. Accordingly,

$$n+1<\frac{n+1}{n}\,\frac{1}{\log(s-1)}\,e^{r^{\rho+\varepsilon}_{p-2}} \leqq \frac{2}{\log(s-1)}\,e^{r^{\rho+\varepsilon}_{p-2}} < \frac{2}{\log(s-1)}\,e^{(sr_{n+1})^{\rho+\varepsilon}_{p-2}},$$

from which it follows that

---

[1] L'exposant de convergence ρ de la suite des r_n est au plus égal à ρ'. (Borel, *loc. cit.*, p. 74.)

$$(n+1)^{1+\varepsilon'} \leqq \left\{\frac{2}{\log(s-1)}\right\}^{1+\varepsilon'} e^{r_{n+1}^{\rho+\varepsilon''}_{p-2}} \qquad (\varepsilon'>0)$$

where $\varepsilon''(>\varepsilon)$ is a certain positive value which can be made as small as we please by taking R sufficiently great and ε', sufficiently small. But $\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon'}}$ is convergent and so also for $\sum_{n=1}^{\infty} \frac{1}{e^{r_n^{\rho+\varepsilon''}}_{p-2}}$, so that

$$\omega^{p'}_{\cdot\cdot}{}^{-2}\rho' \leqq \omega^{p-2}(\rho+\varepsilon''),$$

which holds for any positive value ε''. Hence we have

$$\omega^{p'}_{\cdot\cdot}{}^{-2}\rho' \leqq \omega^{p}_{\cdot\cdot}{}^{-2}\rho,$$

proving the proposition.

5. *Theorem 4. Extension of Hadamard's second theorem*[1] *to entire functions of the transfinite orders.*

Let

$$f(z) \equiv z^{\lambda} \prod_{n=1}^{\infty} \left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

be an entire function of the order $\omega^{p}_{\cdot\cdot}{}^{-2}\rho$. *Then, for any positive values* ε (*however small*) *and* G (*however great*) *there is a circle whose radius is greater than* G *and on which*

$$|f(z)| \geqq e^{-e^{r^{\rho+\varepsilon}_{p-2}}}.$$

Let all the annular domains which are expressed by

$r_n - \frac{1}{e^{r_n^{\rho+\varepsilon'}}_{p-2}} < x < r_n + \frac{1}{e^{r_n^{\rho+\varepsilon'}}_{p-2}}$ be excluded from the total z-plane, where ε' is a certain positive number. Determine n_1 as in §2, and for an arbitrarily assigned value of z $(r>r_{n_1})$ in the remaining domain, determine n_2 as in §2, which is necessarily greater than or equal to n_1. Then

$$f(z) \equiv z^{\lambda} \prod_{n=1}^{n_1} \left(1-\frac{z}{a_n}\right) \prod_{n_1+1}^{n_2} \left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}} \times \prod_{n_2+1}^{\infty} \left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

[1] Étant donnés un produit canonique $G(z)$ de facteurs primaires d'ordre ρ et un nombre positif arbitraire ε, on peut touver une infinité de rayons indefiniment croissants sur chacun desquels on a l'inégalité $|G(z)|>e^{-r^{\rho+\varepsilon}}$. (Borel. *loc. cit.*, p. 76).

We now consider $z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)$.

$$z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right) \geqq r^{\lambda}\left(\frac{r}{r_1}-1\right)\left(\frac{r}{r_2}-1\right)\cdots\cdots\left(\frac{r}{r_{n_1}}-1\right)$$

$$\geqq r^{\lambda}\left(\frac{r}{r_{n_1}}-1\right)^{n_1}$$

$$\geqq e^{-r^{\rho+\varepsilon_1}_{p-2}}$$

for $r \geqq R_1$, where ε_1 is any assigned positive value and R_1 is the corresponding value to it. Secondly, we consider

$$\prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

$$\left|\prod_{n_1+1}^{n_2}\right| \geqq \prod_{n_1+1}^{n_2}\left|1-\frac{r}{r_n}\right| e^{-\left\{\frac{r}{r_n}+\frac{1}{2}\left(\frac{r}{r_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{r}{r_n}\right)^{p_n}\right\}}$$

$$\geqq \prod_{n_1+1}^{n_2}\left|1-\frac{r_n \pm \frac{1}{r_n^{\rho+\varepsilon'}}}{r_n}\right| e^{-\left\{\frac{r}{r_n}+\frac{1}{2}\left(\frac{r}{r_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{r}{r_n}\right)^{p_n}\right\}}$$

$$> \prod_{n_1+1}^{n_2}\frac{1}{r_n e^{r_n^{\rho+\varepsilon'}}_{p-2}} e^{-p_n r^{p_n}}$$

$$\geqq \left\{\frac{1}{r_{n_2} e^{r_{n_2}^{\rho+\varepsilon'}}_{p-2}} e^{-p_k r^{p_k}}\right\}^{n_2-n_1}$$

$$\geqq e^{-n_2\left\{\log r_{n_2} + e^{r_{n_2}^{\rho+\varepsilon'}}_{p-3} + p_k r^{p_k}\right\}},$$

where p_k is the greatest of $p_{n_1+1}, p_{n_1+2}, \ldots\ldots, p_{n_2}$. But we have, similarly as in §3,

$$p_k < (\log k)^{1+\varepsilon''} \leqq (\log n_2)^{1+\varepsilon''}$$

where ε'' is a positive value which can be made as small as we please, by taking n_2 sufficiently great. Hence, by similar reasoning as in §3, we have

$$\left|\prod_{n_1+1}^{n_2}\right| \geqq e^{-r^{\rho+\varepsilon_2}_{p-2}}$$

for $r \geqq R_2$, where ε_2 is any prescribed positive value and R_2 is the corresponding value to it. We consider lastly

$\prod_{n_2+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$. By repeating the process carried out in §2, we have

$$\left|\prod_{n_2+1}^{\infty}\right| > \prod_{n_2+1}^{\infty} e^{-\frac{\left(\frac{r}{r_n}\right)^{p_n+1}}{1-\frac{r}{r_n}}} > e^{-\frac{A}{1-e^{-\frac{1}{E}}}},$$

and accordingly

$$\left|\prod_{n_2+1}^{\infty}\right| \geqq e^{-e^{r^{\rho+\varepsilon_3}}_{p-2}}$$

for $r \geqq R_3$, where ε_3 is any assigned positive value and R_3 is the corresponding value to it. Consequently we have

$$\left|f(z)\right| \equiv \left|z^{\lambda}\prod_{n=1}^{n_1}\cdot\prod_{n_1+1}^{n_2}\cdot\prod_{n_2+1}^{\infty}\right|$$

$$\geqq e^{-e^{r^{\rho+\varepsilon_1}}_{p-2}}\cdot e^{-e^{r^{\rho+\varepsilon_2}}_{p-2}}\cdot e^{-e^{r^{\rho+\varepsilon_3}}_{p-2}}$$

for $r \geqq R_1$, R_2, R_3, or

$$\geqq e^{-e^{r^{\rho+\varepsilon}}_{p-2}}$$

for $r \geqq R$, where ε is assigned positive value and R is the corresponding value to it.

We have excluded all the annular domains expressed by

$$r_n - \frac{1}{e^{r_n^{\rho+\varepsilon'}}_{p-2}} < x < r_n + \frac{1}{e^{r_n^{\rho+\varepsilon'}}_{p-2}}, \qquad (n=1, 2, \cdots\cdots)$$

from the total z-plane and have considered the points in the remaining domain. But the sum of all the annular domains

$$\leqq \pi\sum_{n=1}^{\infty}\left\{\left(r_n+\frac{1}{e^{r_n^{\rho+\varepsilon'}}_{p-2}}\right)^2-\left(r_n-\frac{1}{e^{r_n^{\rho+\varepsilon'}}_{p-2}}\right)^2\right\}$$

$$= 4\pi\sum_{n=1}^{\infty}\frac{r_n}{e^{r_n^{\rho+\varepsilon'}}_{p-2}}.$$

For any positive value $\sigma < \varepsilon'$ there corresponds a positive integer N such that

$$\frac{r_n}{e^{r_n^{\rho+\varepsilon'}}_{p-2}} \leqq \frac{1}{e^{r_n^{\rho+\sigma}}_{p-2}} \qquad \text{for} \qquad n > N.$$

As $\omega^{p'-2}\rho' \leqq \omega^{p-2}\rho$ (§4), $\sum_{n=N}^{\infty} \frac{1}{r_n^{\rho+\sigma}}$ is convergent and so also for $\sum_{n=1}^{\infty} \frac{r_n}{r_n^{\rho+\varepsilon'}}$; so that the sum of all the annular domains is finite and there exists a circle whose radius is greater than G and on which $|f(z)| \geqq e^{-r^{\rho+\varepsilon}}$.

Corollaly 1. *Let* $f_1(z), f_2(z), \ldots\ldots f_k(z)$, *be the canonical products of the orders* $\omega^{p^{(1)}-2}\rho_1$, $\omega^{p^{(2)}-2}\rho_2$, ..., $\omega^{p^{(k)}-2}\rho_k$ *respectively. Then for any prescribed positive value* ε, *there is a circle whose radius is greater than any prescribed positive value* G *and on which*

$$|f_i(z)| \geqq e^{-r^{\rho_i+\varepsilon}}, \qquad (i=1, 2, \ldots\ldots, k).$$

Let the moduli of the zero points of $f_i(z)$ be $r_1^{(i)}, r_2^{(i)}, \ldots, r_n^{(i)}, \ldots\ldots$

Then, we have, by the theorem, $|f_i(z)| \geqq e^{-r^{\rho_i+\varepsilon}}$ for the values of r which are greater than G and satisfy

$$r_n^{(i)} + \frac{1}{r_n^{(i)\rho_i+\varepsilon'}} \leqq r \leqq r_{n+1}^{(i)} - \frac{1}{r_{n+1}^{(i)\rho_i+\varepsilon'}}$$

for a certain value of n. But $\sum_{i=1}^{k}\sum_{n=1}^{\infty} \frac{r_n^{(i)}}{r_n^{(i)\rho_i+\varepsilon'}}$ being finite, the sum of all the annular domains is finite and there exists a circle whose radius is greater than G and on which

$$|f_i(z)| \geqq e^{-r^{\rho_i+\varepsilon}}, \qquad (i=1, 2, \ldots\ldots k).$$

Corollaly 2. *Let*

$$F(z) \equiv e^{F_1(z)} \cdot f_0(z)$$

$$F_1(z) \equiv e^{F_2(z)} \cdot f_1(z)$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$F_{p-2}(z) \equiv e^{F_{p-1}(z)} \cdot f_{p-2}(z),$$

where $F_{p-1}(z)$ *is a polynominal of the* ρ^{th} *degree and* $f_i(z)$, $(i=0, 1, \ldots, p-2)$, *is a canonical product of the order* $\omega^{p^{(i)}-(2+i)}\rho_i$. *If* $\omega^{p-2}\rho$ *be greater*

than $\omega^{p-2}_{\cdot\cdot}{}^{(i)}\rho_i$, $(i=0, 1, \ldots, p-2)$, *the same will be true of the order of* $F(z)$.

For any prescribed positive value ε, there exists, by corollary 1, a circle whose radius is greater than any prescribed positive value and on which

$$|f_i(z)| \geqq e^{-e^{r^{\rho_i+\varepsilon}}_{p^{(i)}-(2+i)}}, \qquad (i=0, 1, \ldots\ldots, p-2).$$

As $F_{p-1}(z)$ is a polynominal of the ρ^{th} degnee, $e^{F_{p-1}(z)}$ is a *regularly* increasing entire function of the ρ^{th} order, that is, the maximum value of $\left|e^{F_{p-1}(z)}\right|$ for $|z|=r$ lies between $e^{r^{\rho-\varepsilon}}$ and $e^{r^{\rho+\varepsilon}}$ for sufficiently great r. Hence, we have, on the circle

$$\max\left|e^{F_{p-1}(z)}\cdot f_{p-2}(z)\right| \geqq e^{r^{\rho-\varepsilon}}\cdot e^{-e^{r^{\rho+\varepsilon}_{p-2}}_{p-p^{(p-2)}}} \geqq e^{r^{\rho-\varepsilon_1}},$$

since $\omega^{p-2}_{\cdot\cdot}\rho > \omega^{p-2}_{\cdot\cdot}{}^{(p-2)}\rho_{p-2}$. As $\left|e^{e^{F_{p-2}(z)}_{\cdot}} f_{p-2}(z)\right|$ and $e^{\left|e^{F_{p-2}(z)}_{\cdot} f_{p-2}(z)\right|}$ are of the same orders, we have, similarly,

$$\max\left|e^{e^{F_{p-1}(z)}_{\cdot}} f_{p-2}(z)\cdot f_{p-3}(z)\right| \geqq e^{e^{r^{\rho-\varepsilon_1}}_{\cdot}\cdot e^{-e^{r^{\rho+\varepsilon}_{p-3}}_{p-(p-1)^{(p-3)}}}} \geqq e^{e^{r^{\rho-\varepsilon_2}}}$$

on that circle, since $\omega^{p-2}_{\cdot\cdot}\rho > \omega^{p-2}_{\cdot\cdot}{}^{(p-3)}\rho_{p-3}$. By repeating the same reasoning, we have

$$\max|F(z)| \equiv M(r) \geqq e^{r^{\rho-\varepsilon_{p-1}}_{p-1}}$$

on the assigned circle, where ε_{p-1} is a certain positive value which becomes as small as we please for sufficiently great r. Determine r so large that $\omega^{p-2}(\rho-\varepsilon_{p-1})$ becomes greater than the greatest of $\omega^{p-2}_{\cdot\cdot}{}^{(i)}\rho_i$ $(i=0, 1, \ldots\ldots, p-2)$. Then the order of $F(z)$, being greater than or equal to $\omega^{p-2}(\rho-\varepsilon_{p-1})$, is greater than the greatest of $\omega^{p-2}_{\cdot\cdot}{}^{(i)}\rho_i$, $(i=0, 1, \ldots\ldots, p-2)$, which is the proposition.

Corollary 3. *If*

$$F(z)\equiv e^{Q(z)}_{\cdot}\, z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

be of the $\omega^{p-2}_{\cdot\cdot}\rho^{th}$ *order,* $e^{Q(z)}$ *will be, at most, of the same order.*

Suppose that the order $\omega^{p'-2}_{\cdot\cdot}\rho'$ of $e^{Q(z)}$ be higher than $\omega^{p-2}_{\cdot\cdot}\rho$, and determine two positive value ε and ε' such that

$$\omega^{p'-2}_{\cdot\cdot}(\rho'-\varepsilon') > \omega^{p-2}(\rho+\varepsilon).$$

Then there exists a positive value R_1 corresponding to ε, such that

$$|F(z)| \leqq e^{r^{\rho+\varepsilon}_{p-1}} \qquad \text{for} \qquad r \geqq R_1.$$

By theorems 2, 3 and 4, we have

$$\left| z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots\cdots + \frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}} \right| > e^{-r^{\rho+\varepsilon}_{e_{p-2}}}$$

for any z $(r \geqq R_2)$ in the exterior of the annular domains

$$r_n - \frac{1}{r_n^{\rho+\varepsilon}_{e_{p-2}}} < x < r_n + \frac{1}{r_n^{\rho+\varepsilon}_{e_{p-2}}}, \qquad (n = 1, 2, \cdots\cdots),$$

and, accordingly, we have $\left| e^{Q(z)} \right| < e^{r^{\rho+\varepsilon}_{2e_{p-2}}}$ for z $(r \geqq R_1, R_2)$ in the same region. The order of $e^{Q(z)}$ however being $\omega^{p'-2}\rho'$,

$$e^{r^{\rho'-\varepsilon'}_{p'-1}} \leqq \left\{ \text{the maximum value of } \left| e^{Q(z)} \right| \text{ for } |z| = r \right\} \leqq e^{r^{\rho'+\varepsilon'}_{p'-1}}$$

must be satisfied for infinitely many values of z which diverges to infinity, and from what has been proved, they must be in the interior of the annular domains. The maximum value of $|e^{Q(z)}|$ for $|z| = r$ increasing with r, we have

$$\left(r_n - \frac{1}{r_n^{\rho+\varepsilon}_{e_{p-2}}}\right)^{\rho'-\varepsilon'}_{e_{p'-1}} \leqq e^{r^{\rho'-\varepsilon'}_{p'-1}}$$

$$\leqq \text{the maximum value of } |e^{Q(z)}| \text{ for } |z| = r$$

$$\leqq \text{the maximum value of } |e^{Q(z)}| \text{ for } |z| = r_n + \frac{1}{r_n^{\rho+\varepsilon}_{e_{p-2}}}$$

$$\leqq e^{2e_{p-2}\left(r_n + \frac{1}{r_n^{\rho+\varepsilon}_{e_{p-2}}}\right)^{\rho+\varepsilon}},$$

where $$r_n - \frac{1}{r_n^{\rho+\varepsilon}_{e_{p-2}}} < r < r_n + \frac{1}{r_n^{\rho+\varepsilon}_{e_{p-2}}}.$$

This is however impossible for sufficiently great n, since $\omega^{p'-2}(\rho' - \varepsilon') > \omega^{p-2}(\rho + \varepsilon)$ and $\lim_{n=\infty} \frac{1}{r_n^{\rho+\varepsilon}_{e_{p-2}}} = 0$. Hence, the proposition is true.

Corollary 4. *Let*

$$F(z) \equiv e^{F_1(z)} \cdot f_0(z)$$

$$F_1(z) \equiv e^{F_2(z)} \cdot f_1(z)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$F_{p-2}(z) \equiv e^{F_{p-1}(z)} \cdot f_{p-2}(z),$$

where $F_{p-1}(z)$ *is a polynominal and* $f_i(z)$, $(i = 0, 1, \ldots, p-2)$, *is a canonical product of the order* $\omega^{p-(2+i)}_{\bullet}\rho_i$. *Then* $\omega^{p-2}_{\bullet}\rho_i$, $(i = 0, 1, \ldots, p-2)$, *is at most equal to the order of* $F(z)$.

Let the order of $F(z)$ be $\omega^{p-2}\rho$. Then by theorems 2 (§3) and 3 (§4), the order $\omega^{p-2}\overset{(0)}{\rho}_0$ of $f_0(z)$ is, at most, equal to $\omega^{p-2}\rho$. As $e^{F_1(z)}$ is, by corollary 3, at most of the order $\omega^{p-2}\rho$, $F_1(z)\equiv e^{F_2(z)}\cdot f_1(z)$ is at most of the $\omega^{p-(2+1)}\rho^{th}$ order. Hence, again by theorems 2 and 3, the order $\omega^{p-(2+1)}\overset{(1)}{\rho}_1$ of $f_1(z)$ is not greater than $\omega^{p-(2+1)}\rho$, or $\omega^{p-2}\overset{(1)}{\rho}_1\equiv\omega^{p-2}\rho$. We have similarly by the alternate applications of theorems 2 and 3 and of corollary 3,

$$\omega^{p-2}\overset{(i)}{\rho}_i \leqq \omega^{p-2}\rho \qquad (i=0, 1, \ldots\ldots, p-2).$$

6. *Theorem 5. If the order of infinitude of* $r_n=|a_n|$, $(n=1, 2, \ldots\ldots)$, *be determinate, the function.*

$$F(z)\equiv e^{Q(z)}z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

will be increasing regularly.

We assume that $F(z)$ is increasing irregularly, and that the exponent of convergency of r_n, $(n=1, 2, \ldots\ldots)$, is equal to the order of $F(z)$[1]. Supposing that ε is any prescribed positive value, and that R is the corresponding value, we have by theorem 2 (§3),

$$|F(z)| \leqq e^{r^{\rho+\varepsilon}_{p-1}} \qquad \text{for all} \qquad r \geqq R.$$

As $F(z)$ is, by assumption, increasing irregularly, there is a number $\omega^{p'-2}\sigma$, finite or transfinite, such that $\omega^{p-2}\rho > \omega^{p'-2}\sigma$ and $|F(z)| \leqq e^{r^{\sigma}_{p'-2}}$ for infinitely many values of r which increase without limit. Supposing that $|z|=r$ be one of such values, determine a positive integer n as to satisfy

$$sr_n \leqq r < sr_{n+1},$$

where s is a prescribed positive integer greater than 2. Then we have, by the lemma in §4.

$$n\log(s-1) < \log M(r) < e^{r^{\sigma}_{p'-2}} < e^{(sr_n+1)^{\sigma}_{p'-2}},$$

or

$$(12) \qquad r_{n+1} > \frac{\{\log_{p'-2}(n\log(s-1))\}^{\frac{1}{\sigma}}}{s} > \{\log_{p'-2}(n+1)\}^{\frac{1}{\sigma}-\varepsilon'},$$

where ε' may be taken as small as we please for sufficiently great n. The inequality (12) holds for infinitely many values of n. Supposing that those values be $n_1 < n_2 < \ldots\ldots < n_i < \ldots\ldots$, we have

$$\lim_{i=\infty}\frac{\log_{p'-1}n_i}{\log r_{n_i}} < \frac{\sigma}{1-\sigma\varepsilon'},$$

[1] The assumption is legitimate by the generalized theorem of Picard in §7, Chap. II., the proof of which is independent of the above theorem.

that is, the order of infinitude of r_n $(n=1, 2, \ldots\ldots)$, is indeterminate, proving the theorem.

Theorem 6. Let

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{r_n}\right)^{p_n}}$$

be of the order $\omega^{p-2}\rho$. *If the function* $f(z)$ *be increasing regularly, the order of infinitude of* $r_n=|a_n|$ *will be determinate.*

We assume that $p\geqq 3$[1]. Supposing that the order of infinitude of r_n is indeterminate, let

$$\frac{\log_{p-1}n}{\log r_n}<\sigma \tag{13}$$

for infinitely many integral values of n, where σ is subjected to the condition that $\sigma<\rho$. Suppose that h $(>n_1)$ be an integer which satisfies (13), where n_1 is determined as in §2, and put

$$(\log_{p-2}h)^{\frac{1}{s}}=r, \tag{14}$$

where s is subjected to the condition that

$$\sigma<s<\rho.$$

$$z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

$$=z^{\lambda}\prod_{n=1}^{n_1}\cdot\prod_{n_1+1}^{h-1}\cdot\prod_{h}^{\infty}\cdot$$

$z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)$ being a polynominal of the $(n_1+\lambda)^{th}$ degree,

$\left|z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)\right|<e^{r^s}_{p-1}$ for sufficiently great r. We have, similarly as in §3,

$$\left|\prod_{n_1+1}^{h-1}\right|<e^{h\{1+(\log h)^{1+\varepsilon'}\}e^{(\log h)^{1+\varepsilon'}\log r}},$$

or in virtue of (14),

$$\left|\prod_{n_1+1}^{h-1}\right|\leqq e^{r^{(1+\varepsilon'')s}}_{p-1},$$

where ε'' is a certain positive value which may be taken as small as we please for sufficiently great h.

$$\left|\prod_{n=h}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}\right|$$

$$\leqq\prod_{n=h}^{\infty}e^{\frac{1}{p_n+1}\left(\frac{r}{r_n}\right)^{p_n+1}+\frac{1}{p_n+2}\left(\frac{r}{r_n}\right)^{p_n+2}+\ldots\ldots\ldots}.$$

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[1] For $p=2$, see Borel, *loc. cit.*, p. 110.

[2] By theorem 3 (§4), the exponent of convergency $\omega^{p'-2}\rho'<\omega^{p-2}\rho$.

We have, however, from (13) and (14)

$$r_h^{\sigma} > \log_{p-1} h = r^s.$$

As $\sigma < s$, we have $\frac{\sigma}{s} < 1 - \frac{1}{\log_{p-1} h}$ for sufficiently great h, and accordingly $r < r_h^{\frac{\sigma}{s}} < r_h^{1-\frac{1}{\log_{p-1} h}}$, which satisfies the inequality (6) in §2. Hence we have, as in §2, $\left|\prod_{n=h}^{\infty}\right| < e^{\overline{1-e^{-\frac{1}{B}}}}$ where A and B are certain finite values such that $\sum_{n=h}^{\infty}\left(\frac{r}{r_n}\right)^{p_n+1} = A$ and $\frac{\log_{p-1} n}{\log r_n} < B$ for $n \geqq h$, so that $\left|\prod_{n=h}^{\infty}\right| < e_{p-1}^{r^s}$ for sufficiently great r. We have therefore

$$\left|\prod_{n=1}^{\infty}\right| < e_{p-1}^{r^s} \cdot e_{p-1}^{r^{(1+\varepsilon')s}} \cdot e_{p-1}^{r^s}$$

$$\leqq e_{p-1}^{r^{s+\varepsilon_1}} \tag{15}$$

where ε_1 is a certain positive value which diminishes with $\frac{1}{r}$. As there are infinitely many values of h which satisfy (13), the relation (15) holds for infinitely many values of r which increase without limit, that is, $f(z)$ is increasing irregularly, which is the proposition.

Chapter II.

TRANSCENDENTAL ALGEBRAIC FUNCTIONS.

INTRODUCTION.

Supposing that the zero points of a transcendental integral function $P_i(z)$, $(i=1, 2, \ldots\ldots, k)$, be $z=a_{i1}, a_{i2}, \ldots\ldots, a_{ik}, \ldots\ldots, b_1, b_2, \ldots\ldots, b_k, \ldots\ldots$ where b_k's are the common zero ponts of $P_1(z)$, $P_2(z)$, $\ldots\ldots$, $P_k(z)$, put

$$f(z) \equiv \prod_{n=1}^{\infty}\left(1-\frac{z}{b_n}\right) e^{\frac{z}{b_n}+\frac{1}{2}\left(\frac{z}{b_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{b_n}\right)^{p_n}}{}^{1}$$

and

$$f_i(z) \equiv \prod_{n=1}^{\infty}\left(1-\frac{z}{a_{in}}\right) e^{\frac{z}{a_{in}}+\frac{1}{2}\left(\frac{z}{a_{in}}\right)^2+\ldots\ldots+\frac{1}{p_n^{(i)}}\left(\frac{z}{a_{in}}\right)^{p_n^{(i)}}}, (i=1, 2, \ldots\ldots, k),$$

[1] If the number of b_n's be finite, say N, $f(z) \equiv \prod_{n=1}^{N}\left(1-\frac{z}{b_n}\right)$. Similarly for $f_i(z)$, $(i=1, 2, \ldots\ldots, k)$.

in which p_n's and $p_n^{(i)}$'s are determined by the condition (5) in §2, chap. I. Put

$$P_i(z) \equiv e^{\overline{P}_i(z)} f(z) \cdot f_i(z), \qquad (i = 1, 2, \ldots\ldots, k).$$

where $\overline{P}_i(z)$, $(i = 1, 2, \ldots\ldots, k)$, is evidently an entire function. If the orders of $e^{\overline{P}_i(z) - \overline{P}_1(z)}$, $(i = 2, \ldots\ldots, k)$, be all less than the greatest of all orders of $e^{\overline{P}_i(z)}$, $(i = 1, 2, \ldots\ldots, k)$[1], put

$$K(z) \equiv e^{\overline{P}_1(z)} f(z)$$

and in the other case, put

$$K(z) \equiv f(z).$$

Definition. When $K(z)$ is a constant, we say that $P_1(z)$, $P_2(z)$, ..., $P_k(z)$ are *relative prime.*

Definition. A function $\omega(z)$ which is defined by

$$(1) \qquad P_0(z)\omega^n + P_1(z)\omega^{n-1} + \ldots\ldots + P_n(z) = 0,$$

where n is a positive integer and $P_0(z)$, $P_1(z)$, ..., $P_n(z)$ are transcendental integral function of z, is called a *transcendental algebraic function* of z.

Definition. When the left hand side of (1) be not decomposable into factors of the same form, and $P_0(z)$, $P_1(z)$, ..., $P_n(z)$ be relative prime, (1) is called an *irreducible equation.*

Definition. In an irreducible equation (1), if $\omega^{p-2}\rho$ be the highest of the orders of $P_0(z)$, $P_1(z)$, ..., $P_n(z)$, (1) is called a *transcendental algebraic equation of the* $\omega^{p-2}\rho^{th}$ *order and the* n^{th} *degree* and the function defined by (1) is called a *transcendental algebraic function of the* $\omega^{p-2}\rho^{th}$ *order and the* n^{th} *degree.*

1. *Theorem 1. A transcendental algebraic function behaves algebraically in any finite part of the plane, the number of the branches being constant; and conversely.*

We may assume, without loss of generality, that (1) is an irreducible equation. Then the function $\omega(z)$ defined by (1) is evidently n-valued. As $P_0(z)$, $P_1(z)$, ..., $P_n(z)$ behave regularly in any finite domain D, $\omega(z)$ behaves algebraically in the neighbourhood of z in D, at which $P_0(z)$ and the discriminant $\mathfrak{D}(z)$ of (1) do not vanish. The zero points of $P_0(z)$ in D are finite in number, and at those points, at least one of the branches of $\omega(z)$ becomes infinity, the infinities, how-

[1] This occurs only when all $e^{\overline{P}_i(z)}$, $(i = 1, 2, \ldots\ldots, k)$, are of the same orders.

ever, being of finite orders, since $P_0(z), P_1(z), \ldots\ldots, P_n(z)$ behave like polynominals in D. The equation (1) being irreducible, $\mathfrak{D}(z)$ does not vanish identically in D, and it is an entire function. Accordingly, the zero points of $\mathfrak{D}(z)$ in D are finite in number and so also for the branch points of $\omega(z)$. Moreover, as $P_0(z), P_1(z), \ldots\ldots, P_n(z)$ behave regularly in D, $\omega(z)$ behaves algebraically in the vicinities of the branch points. In short, $\omega(z)$ has, at most, only a finite number of singular points in D, and the singularities are poles, branch points, or the combinations of them, that is, $\omega(z)$ behaves algebraically in any finite domain. Conversely, suppose that $\omega(z)$ is n-valued and behaves algebraically in any finite domain. Then any symmetric polynominal of the branches $\omega_1, \omega_2, \ldots\ldots, \omega_n$ of $\omega(z)$, being one-valued, has, as singularities, only a finite number of poles in any finite domain, so that it is a meromorphic function of z. Accorningly, $\omega(z)$ is the solution of

$$\prod_{i=1}^{n}(\omega-\omega_i)\equiv\omega^n+R_1(z)\omega^{n-1}+\ldots\ldots+R_n(z)=0,$$

where $R_1(z), R_2(z), \ldots\ldots, R_n(z)$ are meromorphic functions of z. Hence, putting $R_1(z)\equiv\frac{P_1(z)}{P_0(z)}, \ldots\ldots, R_n(z)\equiv\frac{P_n(z)}{P_0(z)}$ where $P_0(z), P_1(z), \ldots\ldots, P_n(z)$ are entire functions of z, we have the proposition.

2. *Theorem 2. Let $\omega(z)$ be a transcendental algebraic function of the $\omega^{p-2}\rho^{th}$ order defined by*

$$P_0(z)\omega^n+P_1(z)\omega^{n-1}+\ldots\ldots+P_n(z)=0. \tag{1}$$

Then for any prescribed positive value ε, there corresponds a finite value R such that

$$|P_0(z)\omega|\leqq e_{p-1}^{r^{\rho+\varepsilon}} \qquad \text{for all } r\equiv|z|\geqq R.$$

Suppose, if possible, there were a sequence of values $z_1, z_2, \ldots\ldots, z_i, \ldots\ldots$ which satisfy $\lim_{i=\infty} z_i=\infty$ and $|P_0(z)\omega|>e_{p-1}^{r^{\rho+\varepsilon}}$. Then by (1)

$$\begin{aligned}|P_0\omega|&\leqq|P_1|+|P_2P_0|\cdot|P_0\omega|^{-1}+\ldots\ldots+|P_nP_0^{n-1}|\cdot|P_0\omega|^{-(n-1)}\\&\leqq|P|\,\frac{1-|P_0\omega|^{-n}}{1-|P_0\omega|^{-1}},\end{aligned}$$

where $|P|$ is the greatest of $|P_1|, |P_2P_0|, \ldots\ldots, |P_nP_0^{n-1}|$. As $|P_0(z_i)\omega(z_i)|>e_{p-1}^{r_i^{\rho+\varepsilon}}$ by assumption, where $r_i=|z_i|$, $(i=1, 2, \ldots\ldots)$, we have

$$\lim_{i=\infty}\frac{1-|P_0(z_i)\omega(z_i)|^{-n}}{1-|P_0(z_i)\omega(z_i)|^{-1}}=1,$$

so that

$$\frac{1-|P_0(z_i)\omega(z_i)|^{-n}}{1-|P_0(z_i)\omega(z_i)|^{-1}} \leqq 2 \qquad \text{for} \quad r_i \geqq R_1.$$

$P_1(z)$, $P_2(z)P_0(z)$, ..., $P_n(z)P_0(z)^{n-1}$ being entire functions at most of the $\omega^{p-2}\rho'^{h}$ order, we have $|P| \leqq \frac{1}{2} e_{p-1}^{r^{\rho+\varepsilon}}$ for $r \geqq R_2$. Then for all z_i such as $r_i \geqq R_1$, R_2, we have

$$e_{p-1}^{r_i^{\rho+\varepsilon}} < |P_0(z_i)\omega(z_i)| \leqq \frac{1}{2} e_{p-1}^{r_i^{\rho+\varepsilon}} \cdot 2 = e_{p-1}^{r_i^{\rho+\varepsilon}}$$

which is a contradiction. Hence we have the proposition.

Corollary 1. *For any positive value* ε, *there corresponds a finite value* R *such that* $\left|\frac{P_n(z)}{\omega(z)}\right| \leqq e_{p-1}^{r^{\rho+\varepsilon}}$ *for all* $r \equiv |z| \geqq R$.

Corollary 2. *Extension of Hadamard's second theorem*[1] *to transcendental algebraic functions.*

Let $\omega(z)$ *be a transcendental algebraic function of the order* $\omega^{p-2}\rho$, *defined by* (1). *Then for any prescribed values* ε (*however small*) *and* G (*however great*), *there will be a circle whose radius is greater than* G *and on which* $|\omega(z)| \leqq e_{p-1}^{|z|^{\rho+\varepsilon}}$. *Similarly, there will be another circle of the same property, on which* $|\omega(z)| \geqq e^{-e_{p-2}^{|z|^{\rho+\varepsilon}}}$.

Supposing that $P_0(z) \equiv e^{P(z)} \cdot f(z)$ where $f(z)$ is the canonical product of the order $\omega^{p'-2}\rho'$ ($\leqq \omega^{p-2}\rho$) of all the zero points of $P_0(z)$, divide both members of (1) by $e^{P(z)}$, by which the order of (1) is invariable. Then we have, by the theorem,

$$|f(z)\omega(z)| \leqq e_{p-1}^{|z|^{\rho+\varepsilon}} \qquad \text{for} \quad |z| \geqq R.$$

By Hadamard's 2^{nd} theorem[2] (for $p'=2$) or by its extension[3] (for $p' \geqq 3$), there is a circle whose radius is greater than any prescribed positive value G, and on which

$$|f(z)| > e^{-e_{p'-2}^{|z|^{\rho'+\varepsilon}}} \geqq e^{-e_{p-2}^{|z|^{\rho+\varepsilon}}}$$

so that

$$|\omega(z)| = |f(z)\omega(z)| \cdot |f(z)|^{-1} \leqq e^{2e_{p-2}^{|z|^{\rho+\varepsilon}}} < e_{p-1}^{|z|^{\rho+\varepsilon'}}$$

where ε' may be taken as small as we please by taking $|z|$ sufficiently great. Thus the first part of the corollary is proved. In precisely the same way, the second part of the corollary may be proved by aid of corollary 1 instead of theorem 2.

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1 See §5, Chap. 1.

2 See foot-note in §5, Chap. 1.

3 See §5, Chap. 1.

3. *Theorem 3. Let*

$$(2) \qquad I_1(z, w) \equiv L_0(z)\omega^{m+n} + L_1(z)\omega^{m+n-1} + \cdots\cdots + L_{m+n}(z),$$
$$(3) \qquad I_2(z, \omega) \equiv M_0(z)\omega^m + M_1(z)\omega^{m-1} + \cdots\cdots + M_m(z),$$
$$(4) \qquad I_3(z, \omega) \equiv N_0(z)\omega^n + N_1(z)\omega^{n-1} + \cdots\cdots + N_n(z),$$

where $L_i(z)$, $(i=0, 1, \ldots\ldots, m+n)$, *are all entire functions and* $M_i(z)$, $(i=0, 1, \ldots\ldots, m)$, *and* $N_i(z)$, $(i=0, 1, \ldots\ldots, n)$, *are generally meromorphic functions. Supposing that*

$$(5) \qquad I_1(z, \omega) \equiv I_2(z, \omega) \cdot I_3(z, \omega),$$

we may determine $\overline{I}_2(z, \omega)$ *and* $\overline{I}_3(z, \omega)$ *such that their degrees in* ω *are equal to those of* $I_2(z, w)$ *and* $I_3(z, \omega)$ *respectively, while their coefficients are entire functions and the functional relation* (5) *still holds.*

Let $M_i(z) \equiv \dfrac{\overline{M}_i(z)}{M(z)}$, $(i=0, 1, \ldots\ldots, m)$, where $M(z)$, $\overline{M}_0(z)$, $\ldots\ldots$, $\overline{M}_m(z)$ are all entire functions and $M(z)$ is the canonical product of the primary factors of the infinity points, at least, of one of $M_i(z)$ (that is to say, in the case that all $M_i(z)$ are rational, $M(z)$ is the least common multiple of their denominators). Similarly, we determine $N(z)$, $\overline{N}_0(z)$, $\overline{N}_1(z)$, $\ldots\ldots$, $\overline{N}_n(z)$. Supposing that α be an arbitrary zero point of $M(z)$, there exists, at least one $\overline{M}_i(z)$ which is indivisible by $(z-\alpha)$. Let $\overline{M}_h(z)$ be the one whose suffix is the least among them. Similarly, if there are $\overline{N}$ which are not divisible by $z-\alpha$, let $\overline{N}_k(z)$ be the first one. Comparing the coefficients of $\omega^{m+n-(h+k)}$ in (5), we have

$$M(z)N(z)L_{h+k}(z) = \overline{M}_0(z)\overline{N}_{h+k}(z) + \overline{M}_1(z)\overline{N}_{h+k-1}(z) + \cdots\cdots + \overline{M}_{h-1}(z)\overline{N}_{k+1}(z) + M_h(z)N_k(z) + \overline{M}_{h+1}(z)N_{k-1}(z) + \cdots\cdots + \overline{M}_{h+k}(z)\overline{N}_0(z),$$

in which all $\overline{M}_i(z)$'s whose suffixes are greater than m, and all $\overline{N}_i(z)$'s whose suffixes are greater than n are zero. Here, on both sides, all terms except $\overline{M}_h(z)\overline{N}_k(z)$ of the right hand are divisible by $(z-\alpha)$, which is impossible. It follows, therefore, that all $\overline{N}_i(z)$'s must be divisible by $(z-\alpha)$. Consequently, α being an arbitrary zero point of $M(z)$, all $\overline{N}_i(z)$'s are divisible by $M(z)$. Similarly all $\overline{M}_i(z)$'s are divisible by $N(z)$. Put

$$\overline{\overline{M}}_i(z) \equiv \frac{M_i(z)}{N(z)}, \qquad (i=0, 1, \ldots\ldots, m),$$
$$\overline{\overline{N}}_i(z) \equiv \frac{N_i(z)}{M(z)}, \qquad (i=0, 1, \ldots\ldots, n).$$

Then $\overline{\overline{M}}(z)$, $(i=0, 1, \ldots\ldots, m)$, and $\overline{\overline{N}}_i(z)$, $(i=0, 1, \ldots\ldots, n)$, are all entire

functions and

$$\overline{I}_2(z, \omega) \equiv \overline{\overline{M}}_0(z)\omega^m + \cdots\cdots + \overline{\overline{M}}_m(z),$$
$$\overline{I}_3(z, \omega) \equiv \overline{\overline{N}}_0(z)\omega^n + \cdots\cdots = \overline{\overline{N}}_n(z)$$

are desired functions.

4. *Theorem 4. Let* $L_i(z)$, $M_i(z)$ *and* $N_i(z)$ *in* (5) *be entire functions and* $M_i(z)$, $(i=0, 1, \ldots\ldots, m)$, *be relative prime.*[1] *If* $\omega^{\rho_1}\dot{-}^2\rho_1$, $\omega^{\rho_2}\dot{-}^2\rho_2$ *and* $\omega^{\rho_3}\dot{-}^2\rho_3$ *be the highest orders among those of* $L_i(z)$, $(i=0, 1, \ldots\ldots, m+n)$, *of* $M_i(z)$, $(i=0, 1, \ldots\ldots, m)$, *and of* $N_i(z)$, $(i=0, 1, \ldots\ldots, n)$, *respectively,* $\omega^{\rho_1}\dot{-}^2\rho_1$ *will be equal to the greater of* $\omega^{\rho_2}\dot{-}^2\rho_2$ *and* $\omega^{\rho_3}\dot{-}^2\rho_3$.

$$(6)\quad \begin{cases} L_0 = M_0 N_0 \\ L_1 = M_0 N_1 + M_1 N_0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ L_i = M_0 N_i + M_1 N_{i-1} + \cdots\cdots + M_i N_0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ L_{m+n} = M_m N_n. \end{cases}$$

As $L_0 = M_0 N_0$ by (6), the canonical products in M_0 and N_0 can not be of higher order than L_0, that is, at most of the $\omega^{\rho_1}\dot{-}^2\rho_1{}^{th}$ order, so that we may put $M_0 \equiv M_0 e^k$ and $N_0 \equiv N_0 \cdot e^{-k}$ respectively where M_0 and N_0 are entire functions at most of the $\omega^{\rho_1}\dot{-}^2\rho_1{}^{th}$ order and k is identically zero or e^k is an entire function of order higher than $\omega^{\rho_1}\dot{-}^2\rho_1$ and has no zero point. Putting $M_i \equiv \overline{M}_i \cdot e^k$ $(i=1, 2, \ldots\ldots, m)$, and $N_i \equiv \overline{N}_i \cdot e^{-k}$, $(i=1, 2, \ldots\ldots, n)$, and solving $\overline{M}_i$ (or $\overline{N}_i$) from $m+n$ equations except the first in (6), $\overline{M}_i$ (or $\overline{N}_i$) may be determined as an algebraic function of $L_1, L_2, \ldots\ldots, L_{m+n}, \overline{M}_0$, and $\overline{N}_0$.

$A_0(L_1, \ldots\ldots, L_{m+n}, \overline{M}_0, \overline{N}_0)\overline{M}_i^s + \cdots\cdots + A_s(L_1, \ldots\ldots, L_{m+n}, \overline{M}_0, \overline{N}_0) = 0$, where s is a positive integer, and all A's are polynominals of $L_1, L_2, \ldots\ldots, L_{m+n}, \overline{M}_0, \overline{N}_0$, that is, entire functions at most of the $\omega^{\rho_1}\dot{-}^2\rho_1{}^{th}$ order. Accordingly we have, by theorem 2 in §2, $|A_0\overline{M}_i| \angle e^{r^{\rho_1+\varepsilon}_{p_1-1}}$ for $r \geqq R$, that is, $A_0\overline{M}_i$ is an entire function at most of the $\omega^{\rho_1}\dot{-}^2\rho_1{}^{th}$ order. Put $A_0 \equiv e^{Q_0} \cdot f_0$ and $\overline{M}_i \equiv e^{Q_1} \cdot f_1$ where Q_0, Q_1 are entire functions and f_0, f_1 are the canonical products of the zero points of A_0 and $\overline{M}_i$ respectively. As $A_0 M_i \equiv e^{Q_0+Q_1} \dot{\cdot} f_0 \cdot f_1$ is at most of the order $\omega^{\rho_1}\dot{-}^2\rho_1$, the same is true, by §4, Chap. 1, of $f_0 \cdot f_1$ and accordingly of f_1. Similarly, as A_0 and $A_0\overline{M}_i$ are at most of the order $\omega^{\rho_1}\dot{-}^2\rho_1$, the same is true by cor. 3 in §5, Chap. 1, of e^{Q_0} and $e^{Q_0+Q_1}$, so also that of e^{Q_1}. Consequently, $e^{Q_1} \cdot f_1 \equiv \overline{M}_i$ is at most of the order $\omega^{\rho_1}\dot{-}^2\rho_1$. But $M_i \equiv e^k \overline{M}_i$, $(i=0, 1,$

[1] See Introduction.

....., m), being relative prime by assumption, we have $k \equiv 0$, so that all M's are at most of the $\omega^{\rho_1 - 2}\rho_1^{th}$ order, and so also for all N's. Hence we have $\omega^{\rho_1 - 2}\rho_1 \geqq \omega^{\rho_2 - 2}\rho_2$ and $\omega^{\rho_1 - 2}\rho_1 \geqq \omega^{\rho_3 - 2}\rho_3$. Now, suppose that $\omega^{\rho_1 - 2}\rho_1 > \omega^{\rho_2 - 2}\rho_2$, $\omega^{\rho_3 - 2}\rho_3$. Then in virtue of $m+n+1$ equations in (6), the orders of $L_0, L_1, \ldots\ldots, L_{m+n}$ are not greater than the greater of $\omega^{\rho_2 - 2}\rho_2$ and $\omega^{\rho_3 - 2}\rho_3$ and accordingly are less than $\omega^{\rho_1 - 2}\rho_1$, which is a contradiction.

5. *Theorem 5. No transcendental algebraic equation is satisfied by a transcendental algebraic function of the higher order.*

Let $\omega(z)$ be a transcendental algebraic function of the $\omega^{\rho_1 - 2}\rho_1^{th}$ order defined by

$$(1) \quad P_0(z)\omega^n + P_1(z)\omega^{n-1} + \ldots\ldots + P_n(z) = 0$$

and

$$(7) \quad Q_0(z)\omega^m + Q_1(z)\omega^{m-1} + \ldots\ldots + Q_m(z) = 0$$

be a transcendental algebraic equation of the $\omega^{\rho_2 - 2}\rho_2^{th}$ order where $\omega^{\rho_2 - 2}\rho_2 < \omega^{\rho_1 - 2}\rho_1$. As (1) is irreducible, in order that (7) be satisfied by $\omega(z)$ defined by (1), it must be decomposable, that is

$$Q_0(z)\omega^m + Q_1(z)\omega^{m-1} + \ldots\ldots + Q_m(z) \equiv \Big(P_0(z)\omega^n + \ldots\ldots + P_n(z)\Big) \times \Big(R_0(z)\omega^{m-n} + R_1(z)\omega^{m-n-1} + \ldots\ldots + R_{m-n}(z)\Big).$$

$Q_0(z), Q_1(z), \ldots\ldots, Q_m(z)$ being relative prime, $R_0(z), \ldots\ldots, R_{m-n}(z)$ are, by §3, entire functions and accordingly by §4, $P_0(z), P_1(z), \ldots\ldots, P_n(z)$ are at most of the order $\omega^{\rho_2 - 2}\rho_2$, that is, of orders lower than $\omega^{\rho_1 - 2}\rho$, which is impossible.

Corollary. No transcendental algebraic function satisfies an irreducible equation of the higher order.

This may be proved in the same way as the above theorem.

6. *Theorem 6. Let $R_i(z)$, $(i = 1, 2, \ldots\ldots, m)$, being different from zero, be a rational or a meromorphic function of a lower order than $e^{K_i(z)}$, where $K_i(z)$ is a polynominal or an entire function of order lower than Ω; let $R_0(z)$ be a rational function (zero being included) or a meromorphic function of a lower order than all $e^{K_i(z)}$, $(i = 1, 2, \ldots\ldots, m)$. Then, in order that*

$$(8) \qquad R_0(z) + R_1(z)e^{K_1(z)} + \ldots\ldots + R_m(z)e^{K_m(z)} \equiv 0,$$

it is necessary that, for each $K_i(z)$, there exists, at least one $K_j(z)$ $(i \neq j)$ such that both $e^{K_i(z)}$ and $e^{K_j(z)}$ are of higher orders than $e^{K_i(z) - K_j(z)}$, and all the sums of those terms relating to one another in such a manner, and accordingly R_0 also, vanish identically.

Supposing that the theorem is untrue, we may assume, then that in (8), no $K_i(z)$ has the property above described; for otherwire we combine all the terms relating to one another in such a manner into one term, by which the coefficients of the combined terms, by assumption, do not all vanish identically. First, we assume that $R_0(z)\equiv 0$. Then we may assume, without loss of generality, that $e^{K_1(z)}$ is one of the lowest orders in $e^{K_i(z)}$, $(i=1, 2, \ldots\ldots, m)$. Dividing both members of (8) by $R_1(z)e^{K_1(z)}$, and differentiating with respect to z, we have

$$(8)' \qquad \sum_{i=2}^{m}\left\{\frac{d\left(\frac{R_i}{R_1}\right)}{dz}+\frac{R_i}{R_1}\frac{d(K_i-K_1)}{dz}\right\}e^{K_i-K_1}\equiv 0$$

of which any coefficient $\dfrac{d\left(\frac{R_i}{R_1}\right)}{dz}+\dfrac{R_i}{R_1}\dfrac{d(K_i-K_1)}{dz}$ does not vanish identically. For if $\dfrac{d\left(\frac{R_i}{R_1}\right)}{dz}+\dfrac{R_i}{R_1}\dfrac{d(K_i-K_1)}{dz}\equiv 0$, we have, by integration, $\dfrac{R_i}{R_1}\equiv Ce^{-(K_i-K_1)}$ where C is a constant different from zero. By the assumption that at least one of e^{K_1} and e^{K_i} is not of a higher order than $e^{K_i-K_1}$ and that e^{K_1} is one of the lowest orders, $e^{K_i-K_1}$ is not of a lower order than e^{K_1}. Suppose that $e^{K_i-K_1}$ be of an order lower than e^{K_i}. As the order of the product of the finite number of entire functions is not greater than all of their orders, the order of $e^{K_i}\equiv e^{K_i-K_1}\cdot e^{K_1}$ is not greater than that of $e^{K_i-K_1}$, and accordingly by assumption, is less than itself, which is a contradiction. Similarly $e^{K_i-K_1}$ is not of a higher order than e^{K_i}. Hence e^{K_i} and $e^{K_i-K_1}$ are of the same orders. By the assumption that R_1 and R_i are of lower orders than e^{K_1} and e^{K_i} respectively, and that e^{K_1} is one of the lowest orders in e^{K_i}, $(i=1, 2, \ldots\ldots, m)$, the order of $\dfrac{R_i}{R_1}$[1] is less than that of e^{K_i}, that is, is less than that of $e^{K_i-K_1}$, and the above identity does not hold. The order of an entire function does not increase by differentiation,[2] and this theorem may easily be extended to mermorphic functions, so that $\dfrac{d}{dz}\left(\dfrac{R_i}{R_1}\right)$ is not of a higher order than $\dfrac{R_i}{R_1}$. Similarly, $\dfrac{d(K_i-K_1)}{dz}$ is not of an order higher than K_i-K_1, which is of a lower order than $e^{K_i-K_1}$. Hence, it follows that $e^{K_i-K_1}$ is of an order higher than

[1] All rational functions are of the order *zero*.

[2] See Borel, *Leçons sur les Fonctions méromorphes*, p. 60.

$\frac{d}{dz}\left(\frac{R_i}{R_1}\right)+\frac{R_i}{R_1}\frac{d(K_i-K_1)}{dz}$. Consequently, we have

$$R_2^{(1)}e^{K_2-K_1}+R_3^{(1)}e^{K_3-K_1}+\cdots\cdots+R_m^{(1)}e^{K_m-K_1}\equiv 0,$$

where $R_i^{(1)}$, being different from zero, is a rational or a meromorphic function of a lower order than $e^{K_i-K_1}$. By repeating the same reasoning, we would have

$$R_m^{(m-1)}e^{K_m-K_{m-1}}\equiv 0,$$

which is impossible, since $R_m^{(m-1)}$ is different from zero.

Secondly, we assume that $R_0(z)\not\equiv 0$. In this case, we may prove the theorem, by dividing, at first, both members of (8) by R_0 whose order is less than those of e^{K_i}, $(i=1, 2, \ldots\ldots, m)$, and then by proceeding in the same way as before.

7. We are now to prove the generalized theorem of Picard. Let

$$(1)\qquad P_0(z)\omega^n+P_1(z)\omega^{n-1}+\cdots\cdots P_n(z)=0,$$

and

$$(9)\qquad Q_0^{(i)}(z)\omega_m+Q_1^{(i)}(z)\omega^{m-1}+\cdots\cdots+Q_m^{(i)}(z)=0,\ (1, 2, \ldots\ldots),$$

be irreducible equations of the $\omega^{p-2}\rho^{th}$ order and the n^{th} degree, and of the $\omega^{p_i-2}\rho_i^{th}$ order and the m^{th} degree respectively. In order that $\omega_i=\omega$ at a certain point, it is necessary and sufficient that

$$\begin{vmatrix} P_0, P_1, \ldots\ldots\ldots, P_n, 0, 0, \ldots\ldots\ldots\ldots, 0 \\ 0, P_0, \ldots\ldots\ldots\ldots, P_n, 0, \ldots\ldots\ldots\ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0, \ldots\ldots, 0, P_0, \ldots\ldots\ldots\ldots\ldots\ldots, P_n \\ Q_0^{(i)}, Q_1^{(i)}, \ldots\ldots\ldots, Q_m^{(i)}, 0, 0, \ldots\ldots\ldots\ldots, 0 \\ 0, Q_0^{(i)}, \ldots\ldots\ldots\ldots\ldots, Q_m^{(i)}, 0, \ldots\ldots\ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0, \ldots\ldots\ldots\ldots\ldots, 0, Q_0^{(i)}, \ldots\ldots\ldots\ldots, Q_m^{(i)} \end{vmatrix} \equiv \psi_{i1}\varphi_1+\psi_{i2}\varphi_2+\cdots\cdots+\psi_{iM}\varphi_M$$

vanishes at that point, where $M={}_{m+n}C_m=\frac{(m+n)!}{m!\ n!}$ and ψ and φ are homogeneous polynominals of the n^{th} degree in $Q^{(i)}$ and of the m^{th} degree in P respectively.

Theorem 7. Picard's theorem generalized.[1]

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[1] Le theorème de M. Picard:—Une fonction entière $F(z)$ telle que les équations $F(z)=a$, $F(z)=b$, $a\neq b$, naient pas de racines, se réduit nécessairement à une constante. (Borel: *Leçons sur les fonctions entières*, p. 88).

Extention aux fonctions méromorphes:—Etant donnée une fonction méromorphe $f(z)$ d'ordre ρ et une autre fonction méromorphe quelconque $\varphi(z)$ d'ordre inférieur, parmi les équations $f(z)=\varphi(z)$ il n'y en a pas en général déxceptionnelles, et s'il en a, il y en a deux an plus. (Borel:—*Leçons sur les fonctions méromorphes*, p. 66).

Let $\dot{\omega}(z)$ *be a transcendental algebraic function of the* $\omega^{p\cdot\cdot-2}\rho^{th}$ *order and the* n^{th} *degree and* $\omega_i(z)$, $(i=1, 2, \ldots\ldots, r)$, *be any transcendental algebraic functions of the* $\omega^{p_i\cdot\cdot-2}\rho_i{}^{th}$ *order* $(\omega^{p_i\cdot\cdot-2}\rho_i<\dot{\omega}^{p\cdot\cdot-2}\rho)$ *and the* m^{th} *degree respectively, under the condition that none of the determinants*

$$(10)\qquad \begin{vmatrix} \psi_{r_1 1} & \psi_{r_2 1} & \ldots\ldots\ldots\ldots & \psi_{r_M 1} \\ \psi_{r_1 2} & \psi_{r_2 2} & \ldots\ldots\ldots\ldots & \psi_{r_M 2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \psi_{r_1 M} & \psi_{r_2 M} & \ldots\ldots\ldots & \psi_{r_M M} \end{vmatrix} \quad \begin{pmatrix} r_1, r_2, \ldots\ldots r_M = 1, 2, \ldots\ldots, r \\ r_1 < r_2 < \ldots\ldots\ldots\ldots < r_M \end{pmatrix},$$

vanishes identically.[1] *Then all the orders of the canonical products of the primary factors corresponding to the points which satisfy* $\omega(z)=\omega_i(z)$, $(i=1, 2, \ldots\ldots, r)$, *can not be inferior to* $\omega^{p\cdot\cdot-2}\rho$, *unless* $r\leqq N=2(M-1)$.

Let (1) and (9) be the equations which define $\dot{\omega}(z)$ and $\omega_i(z)$ respectivly, and $\omega^{q_i\cdot\cdot-2}\sigma_i$ be the order of the canonical product of the primary factors corresponding to the points which satisfy $\omega(z)=\omega_i(z)$. Let $\omega^{p'\cdot\cdot-2}\rho'$ be the greatest of $\omega^{p_i\cdot\cdot-2}\rho_i$ $(i=1, 2, \ldots\ldots, r)$ and $\omega^{q_i\cdot\cdot-2}\sigma_i$ $(i=1, 2, \ldots\ldots, r)$. Then $\omega^{p'\cdot\cdot-2}\rho<\omega^{p\cdot\cdot-2}\rho$. Now we have

$$(11)\qquad \psi_{i1}\varphi_1+\psi_{i2}\varphi_2+\ldots\ldots+\psi_{iM}\varphi_M\equiv R_i e^{K_i},^{2}\quad (i=1, 2, \ldots\ldots, r).$$

where R_i is the canonical product of the primary factors corresponding to the points which satisfy $\omega(z)=\omega_i(z)$, so that it is an entire function of the $\omega^{q_i\cdot\cdot-2}\sigma_i{}^{th}$ order. Accordingly, e^{K_i} is an entire function at most of the $\omega^{p'\cdot\cdot-2}\rho^{th}$ order. Suppose that $r>M$[3] and eliminating $\varphi_1, \varphi_2, \ldots\ldots, \varphi_M$ from any $M+1$ equations, say equations corresponding to $i=1, 2, \ldots\ldots, M+1$, we have

$$(12)\qquad \begin{vmatrix} \psi_{11}, & \psi_{12}, & \ldots\ldots\ldots\ldots, & \psi_{1M}, & R_1 e^{K_1} \\ \psi_{21}, & \psi_{22}, & \ldots\ldots\ldots\ldots, & \psi_{2M}, & R_2 e^{K_2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \psi_{M1}, & \psi_{M2}, & \ldots\ldots\ldots\ldots, & \psi_{MM}, & R_M e^{K_M} \\ \psi_{M+11}, & \psi_{M+12}, & \ldots, & \psi_{M+1M}, & R_{M+1} e^{K_{M+1}} \end{vmatrix} \equiv 0,$$

in which the coefficient $\pm R_i(\psi_{11}\psi_{22}\ldots\ldots\psi_{i-1\,i-1}\psi_{i+1\,i}\ldots\ldots\psi_{M+1\,M})$ of e^{K_i}, by assumption, does not vanish identically and is at most of the $\omega^{p'\cdot\cdot-2}\rho'^{th}$ order. For each e^{K_i} of the $\omega^{p\cdot\cdot-2}\rho^{th}$ order, there exists at least one e^{K_j} $(j\neq i)$, in $e^{K_1}, e^{K_2}, \ldots\ldots, e^{K_{M+1}}$ such that $e^{K_i-K_j}$ is of an order lower

[1] When $m=1$, the condition will be satisfied if $\omega_i(i=1, 2, \ldots\ldots, r)$, are different from one another.

[2] By theorem 5 in §5, $R_i\neq 0$.

[3] If $r\leqq M$, then M being $\geqq 2$, we have $r\leqq M\leqq 2(M-1)=N$.

than those of e^{K_i} and e^{K_j}, or is of an order lower than $\omega^{p-2}\rho$, and accordingly e^{K_j} is of the $\omega^{p-2}\rho^{th}$ order (§6). Now we divide e^{K_1}, e^{K_2}, ..., e^{K_r} into groups such that all e^{K_i}'s which are of order lower than $\omega^{p-2}\rho$ belong to one group, and for other e^{K_i}'s of the order $\omega^{p-2}\rho$, if any two of them have the relation above mentioned, they belong to one and the same group; otherwise, to different groups. Let those groups be G_1, G_2, ..., G_t and s_j be the number of e^{K_i}'s which belong to G_j, ($j=1$, 2, ..., t). Then we hove evidently

$$s_1+s_2+\ldots\ldots+s_t=r. \tag{13}$$

Now all s_j's are less than M. For otherwise, suppose that e^{K_1}, e^{K_2}, ..., e^{K_M} belong to one and the same group. Then as the determinant $(\psi_{11}, \psi_{12}, \ldots\ldots, \psi_{MM})$ by assumption, does not vanish identically, we may express, by solving M equations in (11) corresponding to $i=1, 2, \ldots\ldots, M$, $\varphi_1, \varphi_2, \ldots\ldots, \varphi_M$ as function of the form

$$\varphi_i \equiv R_i e^{K_1}, \quad (i=1, 2, \ldots\ldots, M),$$

where R_i, $(1, 2, \ldots\ldots, M)$, are entire functions of orders lower than $\omega^{p-2}\rho$, and e^{K_1} is an entire function, at most, of the $\omega^{p-2}\rho^{th}$ order. But among $\varphi_1, \varphi_2, \ldots\ldots, \varphi_M$, there exist $P_0^{m-1}P_j$ ($j=0, 1, \ldots\ldots, m$), and we have

$$P_i \equiv S_i e^{\frac{K_1}{m}}, \qquad (i=0, 1, \ldots\ldots, m),$$

where S_i, ($i=0, 1, \ldots\ldots$, m), are entire functions of orders lower than $\omega^{p-2}\rho$. Substituting these values in (1), we have

$$S_0(z)\omega^n + S_1(z)\omega^{n-1} + \ldots\ldots + S_n(z)=0,$$

that is, $\omega(z)$ is of order lower than $\omega^{p-2}\rho$, which is a contradiction. Hence $s_j \leqq M-1$, (j $=1, 2, \ldots\ldots, t$), and consequently we have, from (13),

$$t \geqq 2. \tag{14}$$

Next, the number of all e^{K_i}'s which belong to $G_2, G_3, \ldots\ldots, G_t$ is less than M. For otherwire, taking any $M e^{K_i}$'s, say $e^{K_1}, e^{K_2}, \ldots\ldots, e^{K_M}$, which belong to $G_2, G_3, \ldots\ldots, G_t$ and any e^{K_i}, say $e^{K_{M+1}}$, which belongs to G_1 and eliminating $\varphi_1, \varphi_2, \ldots\ldots, \varphi_M$ from $M+1$ equations in (11) which correspond to $i=1, 2, \ldots\ldots, M+1$, we have the identity (12). Hence there exists at least one e^{K_i} among $e^{K_1}, e^{K_2}, \ldots\ldots, e^{K_M}$, which belongs to the same group as $e^{K_{M+1}}$ (§6). This is however impossible, and accordingly we have

$$s_2+s_3+\ldots\ldots\ldots\ldots+s_t \leqq M-1.$$

Similarly,

$$s_1+s_3+\ldots\ldots\ldots+s_t\leqq M-1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$s_1+s_2+\ldots\ldots\ldots+s_{t-1}\leqq M-1,$$

or, by addition,

$$(t-1)\,(s_1+s_2+\ldots\ldots+s_t)\leqq t(M-1),$$

by (13)

$$r\leqq\frac{t}{t-1}(M-1).$$

As $t\geqq 2$ by (14), we have $\frac{t}{t-1}\leqq 2$ and consequently

$$r\leqq 2(M-1).$$

Theorem 8. If a transcendental algebraic function $\omega(z)$, having only a finite number of branch points, be such that $\omega(z)=a$ and $\omega(z)=b$, $a\neq b$, have no root at finiteness, then it will be an ordinary algebraic function.

Though this theorem is more restricted than the former one, it is very useful for later investigations. As $\omega(z)$ has only a finite number of branch points, any branch of $\omega(z)$ may be expressed as a regular function of $t=z^{\frac{1}{\lambda}}$ in the region $(G\leqq|t|<\infty)$ where λ is a certain positive integer and G, a certain positive value. As $\omega(z)=a$ and $\omega(z)=b$, $a\neq b$, have no root in the assigned region, the infinity point is, by Picard's theorem generalized,[1] a regular or a non-essential singular point of the function $\varphi(t^\lambda)$ of t. Accordingly each of the coefficients $P_0(z)$, ..., $P_n(z)$ in

$$(1)\qquad P_0(z)\omega^n+P_1(z)\omega^{n-1}+\ldots\ldots+P_n(z)=0$$

has, as singularities, at most a pole at infinity, that is, $\omega(z)$ is an ordinary algebraic function of z.

8. We now proceed to consider the Riemann's surface for transcendental algebraic functions. They are entirely analogous to those for ordinary algebraic functions, except for the vicinity of the infinity point. As for the branch lines, we determine them as follows: If $\omega(z)$ be a transcendental algebraic function of the n^{th} degree, the Riemann's surface for it is n-sheeted. If the origin be a branch point, we take as the branch line, a half straight line in any direction, having the origin as its end point, and extending to infinity. For other branch points, $(r_i\theta_i)$,

---

1 Voraussetzung: $f(x)$ sei für $0<|x-x_0|<\rho$ eindeutig-regulär, $\neq a$ und $\neq b$ (wo $a\neq b$ ist).
Behauptung: x_0 ist keine wesentlich singuläre Stelle (sondern regulär oder ein Pol). (E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionen-theorie* p. 96).

we arrange them in order of increasing moduli, and when there are branch points of equal moduli, we arrange them in order of increasing arguments, $(0 \leqq \theta_i < 2\pi)$. Then we have

$$(r_1\theta_1),\ (r_2\theta_2),\ \ldots\ldots,\ (r_i\theta_i),\ \ldots\ldots\ldots .$$

As the branch line corresponding to (r_1, θ_1), we take a half straight line, $R \geqq r_1$, $\theta = \theta_1$, where (R, θ) are the current coordinates, and if there exists another branch point (r_i, θ_i) such that $\theta_i = \theta_1$, we take a small semicircle as a part of the branch line, whose center being (r_i, θ_i), (*see* Fig. 1). For $(r_2\theta_2)$, (r_3, θ_3), ..., similar process will be applicable. As the branch line corresponding to (r_i, θ_i), we take the line determined as above, but slightly deformed so as to have no common point with that corresponding to (r_1, θ_1), and so on. On the Riemann's surface determined as before, $\omega(z)$ is evidently a uniform function of position.

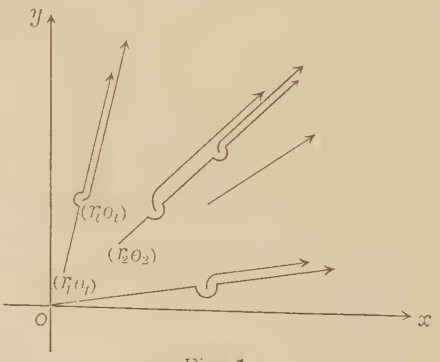

Fig. 1.

9. We shall prove some theorems concerning the Riemann's surface.

Theorem 9. Let S be the Riemann's surface for $\omega(z)$ defined by

$$(1) \qquad P_0(z)\omega^n + P_1(z)\omega^{n-1} + \ldots\ldots + P_n(z) = 0$$

and $f(z, \omega)$ be a uniform function of position on S and be branching out as S. If $f(z, \omega)$ be a holomorphic analytic function of position on the total surface S, the infinity point being included, it will be a constant

Let

$$(a_1, \omega_1),\ (a_2, \omega_2),\ \ldots\ldots,\ (a_n, \omega_n),\ (a_1 = a_2 = \ldots = a_n = a),$$

be n analytic points corresponding to $z = a$. In the vicinity of (a_i, ω_i), $(i = 1, 2, \ldots\ldots, n)$, we have $f(z, \omega) = f_i(z) = f_i(z \mid a)$. $S(f)$, being any symmetric polynominal of $f_1(z), f_2(z), \ldots\ldots, f_n(z)$, has the same value in whatever sheet z may lie and by whatever path z may have attained its position in that sheet. Hence it is a uniform function of z. Moreover, as $S(f)$ is holomorphic on the total surfaces S, the point at infinity being included, the same is true on the total z-plane, the point at infinity being included, and accordingly it is a constant. From this, it follows that, in

$$(f - f_1)(f - f_2)\ldots\ldots(f - f_n) = f^n + S_1 f^{n-1} + \ldots\ldots S_n = 0,$$

all S_i, $(i = 1, 2, \ldots\ldots, n)$, are constants and hence $f(z, \omega)$ is also.

Theorem 10. Let $f(z, \omega)$ be a uniform function of position on S and be branching out as S. If all the singularities be poles, finite in number, in any finite part of S, $f(z, \omega)$ will be rational in ω and meromorphic in z; and conversely.

Let

$$(a_1\omega_1), (a_2\omega_2), \ldots\ldots, (a_n\omega_n), (a_1=a_2=\ldots\ldots a_n=a),$$

be n analytic points corresponding to $z=a$. In the vicinity of $(a_i\omega_i)$, we have $f(z, \omega)=f_i(z\,|\,a)$, $(i=1, 2, \ldots\ldots, n)$. Put

$$(15)\qquad \begin{cases} f_1+f_2+\ldots\ldots f_n=P_0 \\ f_1\omega_1+f_2\omega_2+\ldots\ldots+f_n\omega_n=P_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_1\omega_1^{n-1}+f_2\omega_2^{n-1}+\ldots\ldots+f_n\omega_n^{n-1}=P_{n-1}, \end{cases}$$

where P_i, $(i=0, 1, \ldots\ldots, n-1)$, are evidently one-valued functions of z and have only finite number of poles in any finite part of the z-plane, so that they are mormorphic functions of z. Next, we determine A_{n-1}, A_{n-2}, ..., A_1 so as to satisfy the following relations

$$(16)\qquad \omega_i^{n-1}+A_1\omega_i^{n-2}+\ldots\ldots+A_{n-1}=0, \quad (i=2, 3, \ldots\ldots, n).$$

Multiplying the equations in (15) by A_{n-1}, A_{n-2}, ..., A_1, 1 respectively and adding side by side, we have

$$\sum_{i=1}^{n} f_i(\omega_i^{n-1}+A_1\omega_i^{n-2}+\ldots\ldots+A_{n-1})=P_{n-1}+A_1P_{n-2}+\ldots\ldots A_{n-1}P_0,$$

which is reduced, in virtue of (16), to

$$(17)\quad f_1(\omega_1^{n-1}+A_1\omega_1^{n-2}+\ldots\ldots+A_{n-1})=P_{n-1}+A_1P_{n-2}+\ldots\ldots A_{n-1}P_0.$$

From (16), the roots of $\omega^{n-1}+A_1\omega^{n-2}+\ldots\ldots+A_{n-1}=0$ are $\omega_2, \omega_3, \ldots\ldots, \omega_n$. Hence we have

$$\begin{aligned}\omega^n+\frac{Q_1}{Q_0}\omega^{n-2}+\ldots\ldots+\frac{Q_n}{Q_0} &\equiv(\omega-\omega_1)(\omega-\omega_2)\ldots\ldots(\omega-\omega_n) \\ &\equiv(\omega-\omega_1)(\omega^{n-1}+A_1\omega^{n-2}+\ldots\ldots+A_{n-1}) \\ &=\omega^n+(A_1-\omega_1)\omega^{n-1}+\ldots\ldots-\omega_1A_{n-1}.\end{aligned}$$

Comparing the coefficients, we have

$$A_1=\frac{Q_0\omega_1+Q_1}{Q_0}$$

.....

$$A_i=\frac{Q_0\omega_1^i+Q_1\omega_1^{i-1}+\ldots+Q_i}{Q_0}$$

.....

i.e. A_i, $(i=1, 2, \ldots\ldots, n-1)$, are polynominals of ω_1 whose coefficients are meromorphic functions of z. From (17), we have

$$f_1=\frac{P_{n-1}+A_1P_{n-2}+\ldots\ldots+A_{n-1}P_0}{\omega_1^{n-1}+A_1\omega_1^{n-2}+\ldots\ldots+A_{n-1}}.$$

Proceeding just as above, we may obtain an analogous expression for each branch of ω, so that we have

$$f=\frac{P_{n-1}+A_1P_{n-2}+\ldots\ldots A_{n-1}P_0}{\omega^{n-1}+A_1\omega^{n-2}+\ldots\ldots+A_{n-1}},$$

where $A_i=\dfrac{Q_0\omega^i+Q_1\omega^{i-1}+\ldots\ldots+Q_i}{Q_0}$, $(i=1, 2, \ldots\ldots, n-1)$, i.e. $f(z, \omega)$ is rational in ω and is mermorphic in z.

Conversely, let $f(z, w)$ be a function which is rational in ω and mermorphic in z. In virtue of the equation (1), it may be reduced to the form

$$f(z, w)\equiv\frac{R_0\omega^{n-1}+R_1\omega^{n-2}+\ldots\ldots+R_{n-1}}{S_0\omega^{n-1}+S_1\omega^{n-2}+\ldots\ldots+S_{n-1}},$$

where R_i and S_i $(i=0, 1, \ldots\ldots, n-1)$, are entire functions of z. In the determinant

$$B\equiv\left|\begin{array}{l}
\left.\begin{array}{l} Q_0,\ Q_1,\ \ldots\ldots\ldots\ldots,\ Q_n,\ 0,\ 0,\ \ldots\ldots\ldots\ldots,\ 0 \\ 0,\ Q_0,\ \ldots\ldots\ldots\ldots\ldots\ldots,\ Q_n,\ 0,\ \ldots\ldots\ldots\ldots,\ 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0,\ \ldots\ldots\ldots,\ 0,\ Q_0,\ \ldots\ldots\ldots\ldots\ldots\ldots,\ Q_n \end{array}\right\} n-1 \text{ rows} \\
\left.\begin{array}{l} S_0,\ S_1,\ \ldots\ldots\ldots\ldots\ldots,\ S_{n-1},\ 0,\ 0,\ \ldots\ldots\ldots,\ 0 \\ 0,\ S_0,\ \ldots\ldots\ldots\ldots\ldots,\ S_{n-1},\ 0,\ \ldots\ldots\ldots,\ 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0,\ \ldots\ldots\ldots,\ 0,\ S_0,\ \ldots\ldots\ldots\ldots\ldots\ldots,\ S_{n-1} \end{array}\right\} n \text{ rows},
\end{array}\right|$$

which is evidently an entire function of z, multiply the first $2(n-1)$ columns in order by ω^{2n-2}, ω^{2n-3}, ..., ω and add the products to the last column. Then we have

$$B\equiv\varphi(z, \omega)(S_0\omega^{n-1}+\ldots\ldots+S_{n-1})+\psi(z, \omega)(Q_0\omega^n+Q_1\omega^{n-1}+\ldots\ldots+Q_n),$$

where φ and ψ are polynominals of ω whose coefficients are entire functions of z. But as $Q_0\omega^n+Q_1\omega^{n-1}+\ldots\ldots Q_n\equiv 0$ on S, we have

$$f \equiv \frac{R_0\omega^{n-1}+\cdots\cdots+R_{n-1}}{S_0\omega^{n-1}+\cdots\cdots+S_{n-1}} \cdot \frac{\varphi(z,\omega)}{\varphi(z,\omega)} \equiv \frac{T_0\omega^m+T_1\omega^{m-1}+\cdots\cdots+T_m}{B}$$

where m is a positive integer and T_i $(i=0, 1, \ldots\ldots, m)$, are entire functions of z. Accordingly, it follows from the identity, that $f(z, \omega)$ is one-valued on S and is branching out as S, and moreover it has, as singularities, only a finite number of poles in any finite part of S.

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On the Analysis of Calcium Phosphate.

By

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I. SIMULTANEOUS DETERMINATION OF CALCIUM OXIDE AND PHOSPHORIC ACID.

The usual method of determination of phosphoric acid consists of two procedures: (*a*) the preliminary precipitation of phosphoric acid from an acid solution by means of the ammonium molybdate reagent and (*b*) its final reprecipitation from the ammoniacal solution by means of the magnesia mixture. The new method here proposed for the analysis of calcium phosphate obviates the use of the ammonium molybdate reagent and possesses the advantage of enabling the simultaneous determination of calcium to be carried out.

Calcium is precipitated quantitatively from an acetic acid solution as oxalate in the presence of a measured excess of a standard solution of potassium bioxalate. Calcium oxalate is either titrated with potassium permanganate or ignited and weighed as oxide. The excess of oxalic acid in the filtrate is also titrated with potassium permanganate and the phosphoric acid in the reaction liquid thus obtained is determined by precipitating it as ammonium manganese phosphate.

PRECIPITATION OF CALCIUM OXALATE. 0·5000 gr. of the sample is dissolved in about 10 cc. of water containing 2 cc. of 6 *N* HNO_3. 10 cc. of 6 *N* acetic acid are added and the solution diluted to 200 cc. and heated to near boiling. 30 cc. of 0·2 molar solution of potassium bioxalate[1] are cautiously added drop by drop to the boiling

[1] Y. Osaka and K. Ando, Mem. Coll. Sci., Kyoto, **4**, 371 (1921).

solution with a constant stirring, preferably from a burette or a pipette. 5 cc. of saturated ammonium acetate solution (nearly 6 *N* strong) are then added a few drops at a time with a constant stirring. The mixture is heated 5–10 minutes longer and then neutralized by slowly adding 12–14 cc. of 6 *N* ammonia and then allowed to stand for an hour on a warm plate: the precipitate becomes coarsely crystalline and settles to the bottom of the beaker. The clear liquid is then poured through a filter by decantation. The precipitate is covered with exactly 1 cc. of 2 molar KHC_2O_4 solution and about 10 cc. of boiling water with 1 cc. of 6 *N* ammonia. It is filtered by decantation again and the operation is repeated three times. Finally the whole precipitate is transferred to the filter and washed with hot water. (The total volume of the 2 molar KHC_2O_4 solution used in washing the precipitate is put down in the note-book and all the filtrates and washings are combined and reserved for the determination of phosphoric acid.)

The precipitate is now kept in an air bath until nearly dry, when it is placed together with the filter in a platinum crucible and ignited. The ignition of calcium oxalate is carried out in the ordinary manner, such as described in Treadwell-Hall, Analytical Chemistry, Vol. II. (1915), p. 70.

TITRATION OF THE EXCESS OF POTASSIUM BIOXALATE. The filtrates and washings from calcium oxalate are combined and treated with 1 cc. of a nearly 1 *N* manganous sulphate solution, containing 11·2 per cent. of $MnSO_4 \cdot 4H_2O$. It is warmed to 70° C. and, after addition of 5 cc. of 6 *N* H_2SO_4, titrated with potassium permanganate solution. There will be needed in the present case about 30–32 cc. of 0·1 *N* solution to attain the end reaction.

PRECIPITATION OF AMMONIUM MANGANESE PHOSPHATE. The titrated liquid is concentrated on the water bath to about 200 cc. and then treated with 5 gr. of solid ammonium chloride. It is brought to boiling and there are added to it drop by drop about 5–6 cc. of 6 *N* ammonia with a constant stirring till the liquid smells of it very distinctly. This operation is carried out very slowly because the ammonium manganese phosphate that comes down assumes the crystalline form very much more easily while the liquid is acid with the liberated acetic acid than after it has become alkaline with ammonia. The mixture is heated 5–10 minutes longer and then allowed to stand

for an hour on the warm iron plate with a watch glass over the beaker to prevent the contact of the air as much as possible. The crystalline precipitate is filtered and washed with water containing 5 per cent. of ammonium chloride and 0.2 per cent. of ammonia. It is dried and ignited and weighed in the usual manner.

EXPERIMENTAL RESULTS. The calcium phosphoricum praecipitatum of the Japanese Pharmacopœia was used as the sample for analysis. Heated in the air bath at 160° C. for 5 hours, it lost 13·44 per cent. of water, while theoretically it should have lost 20·91 per cent. of water of crystallisation. Ignited at the red heat, it lost 24·76 per cent. in average, while the theory requires 26·14 per cent. for the total water of $CaHPO_4 \cdot 2H_2O$. The deficit of water was thus 1·38 per cent.

In addition to the fact that the content of water in samples of calcium phosphate is rather irregular, it is also well known that secondary calcium phosphate left in contact with water gives up more or less of its PO_4 ions and becomes richer in calcium in the following sense:

$$4CaHPO_4 \cdot 2H_2O = Ca_3(PO_4)_2 + Ca(H_2PO_4)_2 + 2H_2O.$$

It will be, therefore, very reasonable to assume that the known deficit of water in the sample is due to the above change and that it is compensated by the corresponding amount of tertiary calcium phosphate. Making a calculation based upon this assumption, it is found that the deficit of 1·38 per cent. of water in the sample means that 5·28 per cent. of $CaHPO_4 \cdot 2H_2O$ in it are replaced by the same percentage of $Ca_3(PO_4)_2$. This, furthermore, ultimately amounts to saying that the sample contains, in place of 1·38 per cent. of water in deficit, an excess of CaO to the extent of 1·14 per cent. over the theoretical requirement for pure sample of $CaHPO_4 \cdot 2H_2O$.

There were made altogether thirty seven complete analyses of calcium phosphate for CaO and P_2O_5 with the method here proposed, and most of the results were closely concordant. Many of them were made by Mr. Tamaichi Kawamoto and various other students. The results of four analyses recently made by the author are given here as example:

Substance taken each time: 0·5000 gr.

					average	theory
Percentage of CaO,	33·53,	33·55,	33·48,	33·53,	33·52,	32·59
Percentage of P_2O_5,	41·74,	41·77,	41·84,	41·67,	41·76,	41·27
Percentage of H_2O,	24·75,	24·95,	24·88,	24·70,	24·82,	26·14
					100·10	100·00

The quantity of CaO found is a little in excess of the theoretical value calculated for pure $CaHPO_4 \cdot 2H_2O$, i.e. to an extent of 0·93 per cent. It is worthy of notice that this latter figure is a close approach to the value—1·14—calculated from the deficit of water upon the above hypothesis that the loss of water is compensated by an increaee of CaO to form $Ca_3(PO_4)2$.

REMARK. 1. All the filtrates and washings from the calcium oxalate precipitate may be combined and evaporated to near dryness. In this case, the oxalic acid is expelled by heating the residue with concentrated sulphuric acid and the phosphoric acid determined by precipitation as ammonium magnesium phosphate.

2. In order to be assured that the separation of phosphoric acid is complete, the weighed CaO may be dissolved in HNO_3 and tested with the molybdate reagent in a centrifuge tube. After repeatcd centrifugation and decantation, the precipitate is dried and weighed and the weight of P_2O_5 calculated out. This weight is subtracted from that of CaO and added to that of P_2O_5. The residual P_2O_5 thus reclaimed was found in five experiments to amount to 0·03, 0·08, 0·04, 0·09. and 0·08 per cent. respectively.

II. ACIDIMETRY OF CALCIUM PHOSPHATE.

The primary phosphates are neutral toward methyl orange, while the secondary and tertiary phosphates of alkali metals react alkaline toward it. If a weighed sample of a phosphate is dissolved in a measured excess of 0·1 *N* HCl and titrated back with 0·1 *N* NaOH with methyl orange as indicator, the proportion of the secondary and tertiary salts will be easily calculated out on the basis of the following equations:

$$Ca_3(PO_4)_2 + 4HCl = Ca(H_2PO_4)_2 + 2CaCl_2,$$
$$2CaHPO_4 \cdot 2H_2O + 2HCl = Ca(H_2PO_4)_2 + CaCl_2 + 2H_2O.$$

0·5000 gr. of $Ca_3(PO_4)_2$ contains 1·6115 mg. mols of it and will require $4 \times 16{\cdot}115 = 64{\cdot}46$ cc. of 0·1 *N* acid to form the primary salt, while 0·5000 gr. of $CaHPO_4 \cdot 2H_2O$ contains 2·9045 mg. mols and will require 29·045 cc. of the 0·1 *N* acid. If, therefore, 0·5 gr. of the commercial calcium phosphate was taken for analysis and if one per cent. of it

consisted of the tertiary salt and the remaining 99 per cent. consisted of the secondary salt, then there would be required $29{\cdot}045+(0{\cdot}6446-0{\cdot}2905)=29{\cdot}40$ cc. of 0·1 *N* acid to bring it to solution as the primary salt.

If, therefore, a 0·5000 gr. portion of a sample is thus analysed and V cc. of 0·1 *N* acid found to be consumed to form the primary salt, then the percentage of the tertiary salt in the sample will be indicated by the formula:

$$(V-29{\cdot}05)\div 0{\cdot}354.$$[1]

In the following experiments, 0·5000 gr. portions of a fresh sample of calcium praecipitatum were taken for analysis according to the principle just indicated. The numbers of cc. of 0·1 *N* acid consumed were:

29·44, 29·80, 29·49, 29·60, 29·80; average 29·63 cc.

Applying here the formula given above, the amount of tertiary calcium phosphate in the sample were found to be $(29{\cdot}63-29{\cdot}05)\div 0{\cdot}354=1{\cdot}7$ per cent. As the difference of percentages of CaO in the tertiary and secondary calcium phosphates is $54{\cdot}21-32{\cdot}57=21{\cdot}64$, the above result ultimately amounts to saying that the actual excess of CaO in the analysed sample over that of pure secondary calcium phosphate is $1{\cdot}7\times 21{\cdot}64/100\equiv 0{\cdot}37$ per cent. According to the hypothesis put forward in the first part of this paper, this quantity of CaO has entered the sample as the result of replacing a part of the secondary salt by an equal percentage of the tertiary salt.

III. ALKALIMETRY OF CALCIUM PHOSPHATE.

As I have found by several experiments that secondary calcium phosphate has power to neutralize caustic soda toward phenolphthalein, an investigation was made regarding the possibility of applying this reaction to the determination of the secondary salt in the commercial sample. It may be noted at the outset that there is a marked difference in this respect between an alkali phosphate and an alkaline earth phosphate and that pure disodium phosphate reacts alkaline toward

[1] Here it must be borne in mind that in titrating phosphoric acid with methyl orange an excess of about 0·3 cc. of the tenth-normal alkali is necessary to cause the change from red to yellow. Treadwell-Hall, Analytical Chemistry, Vol. II. (1915), 542.

phenolphthalein while secondary calcium phosphate reacts acid toward it.

When secondary calcium phosphate is left in contact with caustic soda solution for 14 hours, or boiled with it for a few minutes, the latter disappears chiefly according to the following equation:

$$3CaHPO_4 \cdot 2H_2O + 2NaOH = Ca_3(PO_4)_2 + Na_2HPO_4 + 8H_2O.$$

In the following experiments, 0·5000 gr. portions of the same sample that was used in the acidimetry above described were mixed with a measured quantity of 0·1 *N* NaOH, and, after standing for 24 hours with occasional shaking or after boiling for a few minutes, the mixture was filtered and well washed. The filtrate and washings were combined and titrated back with 0·1 *N* HCl with phenolphthalein as indicator. The difference between the two quantities is the quantity of 0·1 *N* NaOH absorbed. The results of four such determinations were:

22·75, 22·80, 22·10, 21·00; average 22·16 cc.

If the reaction had run as required by the above equation so that two molecules of NaOH were consumed by every three molecules of secondary calcium phosphate, then the quantity of NaOH consumed should have been only about 19·5 cc. Thus the above results are by far too discordant to justify any use of this reaction for a quantitative purpose.

Experiments were then started to find out whether a more definite result could be obtained by first dissolving the sample in a measured excess of the deci-normal acid and neutralizing it with deci-normal NaOH toward methyl orange—i.e. bringing it to the stage of the primary salt—and then further titrating it back with phenolphthalein as indicator. It was ascertained beyond any doubt by the results that there is no well defined stage for either the secondary or the tertiary salt during the course of neutralization with NaOH. There was, indeed, a point reached sooner or later when a distinct phenolphthalein color showed up, but this color would invariably fade away in a few minutes to re-appear by addition of a few more drops of NaOH and to fade away again, and so on indefinitely for several hours. By taking as criterion the first transient appearance of the pink color that remained about five minutes, the quantities of deci-normal NaOH required to attain that point were for 0·5000 gr. portions of the sample:

50·80, 48·88, 42·40, 42·72, 48·68; average 46·70 cc.

If the end-point were to be sharply at the stage indicated by the last of the following equations, then the corresponding figure should have been about 49·2 cc. instead of 46·70 cc.:

$$2CaHPO_4 + 2HCl = Ca(H_2PO_4)_2 + CaCl_2,$$
$$Ca(H_2PO_4)_2 + CaCl_2 + 2NaOH = 2CaHPO_4 + 2NaCl + H_2O,$$
$$3CaHPO_4 + 2NaOH = Ca_3(PO_4)_2 + Na_2HPO_4 + 2H_2O.$$

There still remained a possibility to be tested, that in the presense of an excess of calcium chloride the titration of the primary phosphate with NaOH toward phenolphthalein might give a definite end-point in the following sense:

$$2CaHPO_4 + CaCl_2 + 2NaOH = Ca_3(PO_4)_2 + 2NaCl + 2H_2O.$$

Two experiments were made each with 0·5000 gr. of the sample dissolved in a measured excess of deci-normal acid. After bringing the solution to neutrality with deci-normal NaOH toward methyl orange, it was mixed with 20 cc. of 0·6 molar solution of neutral $Ca(NO_3)2$. On titrating it toward phenolphthalein with 0·1 *N* NaOH at a nearly boiling temperatur until the permanent pink color was attained, the number of cc. of the alkali solution required for the above reaction was found to be:

69·70, 70·50, average 70·10 cc.; theoretical 59·00 cc.

The above given series of the alkalimetric titrations clearly shows the impracticability of applying this method to the determination of either secondary or tertiary calcium phosphate in the commercial sample. It also shows clearly the power of tertiary calcium phosphote to absorb, to a more or less extent, calcium oxide as well as caustic soda, probably in the form of solid solution.[1] The fact is well known, on the other hand, that tertiary calcium phosphate left in contact with water loses more PO_4 ions than Ca ions, the resulting solution becoming acid in reaction and the solid phase richer in lime than before.

In view of these results, the present author cannot but help entertaining some grave doubts as to the success of a similar alkalimetric method recently proposed by A. J. Jones[2] for analyzing calcium phosphate of the British Pharmacopœia.

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[1] Moissan, Traité de Chimie Minérale, Tom. 3, 564-566; Seidell, Solubilities, (1916), p. 95; Cameron and Hurst, J. Amer. Chem. Soc., **26**, 903–911 (1904); Cameron and Bell, Ibid., **27**, 1512–1514 (1905); K. K. Jarvinen, Zeits. Anal. Chem., **43**, 559–562 (1904).

[2] A. J. Jones, Phar. J., cviii, 62–63 (1922); quoted in The Analyst, xlvii, 127 (1922).

SUMMARY.

(1) A method of simultaneous determination of calcium and phosphoric acid in calcium phosphate is described. Calcium is precipitated as oxalate from the acetic acid solution and the phosphoric acid in the filtrate is precipitated, after destroying the excess of oxalic acid in it with potassium permanganate, as ammonium manganese phosphate.

(2) The proportion of the secondary and tertiary salts in a sample can also be determined by ascertaining the number of cc. of the deci-normal acid solution consumed in converting a 0·5000 gr. portion of the sample to the stage of the primary phosphate.

(3) An attempt was made to base a method of determination on the results of titrating the solution of primary calcium phosphate, made neutral toward methyl orange, with a standard solution of caustic soda and with phenolphthalein as indicator. This attempt came out futile on account of the absence of any definite end-point for tertiary phosphate due to the fact that the latter absorbs more or less quantity of calcium oxide as a solid solution.

On Mechanical Devices for Demonstration of Vibrations of String and Spring.

By

Shuichi Hayashi.

(Received March 1, 1922.)

(I)

In some cases it is very desirable to have a source of a sound whose pitches can easily be varied. The present writer recently deviced such an instrument and found very convenient both for demonstration and investigation. Fig. 1 shows its cross section, M being a horse-shoe permanent magnet to which two pieces of soft iron P are attached, and C the coil of the magnet. The coil, whose one end is connected to the insulated gas carbon at G, an electric cell and an endless wire W forms a circuit.

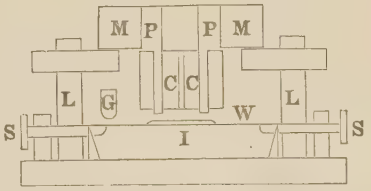


Fig. 1.

In front of the coil a circular iron plate I attached to the wire W is placed, its tension being adjusted by means of two screws at S. The instrument is thus a kind of a modified telephone, so that when an electric current is sent through the circuit, the plate I will vibrate and the sound will be emitted, its pitch depending on the tension of the wire and the mass of the iron plate. Therefore if a set of such wires with different masses is prepared, it serves as a convenient instrument producing sound of various pitches.

With this instrument it is very easy to produce a sound having frequency as high as 5000 per sec., and the amplitude of vibrations may be increased up to about 1/2 mm. when the carbon point was displaced nearer to the leg L.

(II)

To obtain Lissajous' figures in most simple manner, the writer recently deviced an improved vibrator. The cross section of this vibrator is shown in Fig. 2, where C is a coil, S an iron core, H and G are made of soft iron, and E is a spring clamped at D, its length being adjusted by the screw at D, and its distance from the coil, by the screw B. When an intermittent electric current produced by an electrically driven tuning fork is sent through this coil, the spring will vibrate. The amplitude of the vibration will reach a sufficiently larg value when the frequency of the spring just coincides with one of the integral multiples of that of the fork. Thus by adjusting length of the spring and distance from the coil, the vibration having a proper amplitude and a higher frequency will be produced with this simple instrument. With a pair of such vibrators composition of vibrations can easily be effected. Photographs given in Fig. 3, represent composition of vibrations at right angles. Next, the resultant of two parallel vibrations was photographed with the arrangement shown in Fig. 4. S represents a slit, *a* and *b* small mirrors attached to the free ends of the springs of two vibrators described above and facing toward one another, *c* and *d* fixed mirrors placed in the position as indicated in the Fig. 4, the former facing toward *b* and the latter toward *a*. Parallel rays of light passing through the slit are reflected by the system of the mirrors and collected by the lens at L, and form the image of the slit on a moving photographic plate. The rays reflected from a pair of mirrors (*a*, *d*) and from (*b*, *c*) record oscillations of each vibrator, while the image formed by the rays reflected from *a* and *b* gives their resultant vibrations, the light reflected from a set of fixed mirrors (*c*, *d*) giving a straight line on the photographic plate. Photographs shown in Fig. 5 were thus obtained.

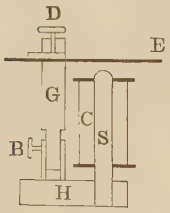


Fig. 2.

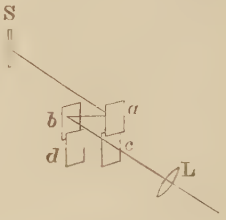


Fig. 4.

(III)

In the case of vibrations of a piano-fort wire strucked by a hammer, it is very difficult to observe initial stages of the configuration of the wire. Kaufmann,[1] in his theoretical treatment of the vibration of a piano-fort wire assumed that every point of the wire starts their periodic vibration as soon as the hammer left the wire. But he was not able to determine the exact instant at which the wire just begins to vibrate. In order to test the above assumption of Kaufmann, the writer recently deviced a special hammer and succeded in photographing initial stages of the vibration at any point of the wire.

The arrangement of the wire and the hammer is shown in Fig. 6. The wire whose position indicated by W is stretched horizontally perpendicular to the plane of the paper and is placed at such a height as it is just struck when the hammer was released by switching off the current in an electromagnet E shown in Fig. 6 and pushed down by a bent spring S. Immediately after the hammer struck the wire, the hinged head of it is held back by means of another spring R and takes the position indicated by the dotted lines. The motion of the wire caused by such an impulse was studied by means of a special camera with a photographic plate moving horizontally, its sections being illustrated in Fig. 7. F is the camera with a vertical slit at S and contains a photographic plate G. A string having a weight W at one end was attached to the plate holder and another string connected to the same weight was fastened to an iron plate in a viscous liquid contained in a vertical tube, both strings hunging over the same pulley P. By this means the horizontal velocity of the photographic plate was regulated. The motion of the plate was started by switching off the current in the electromagnet E which has attracted the iron plate. A is wooden disc having

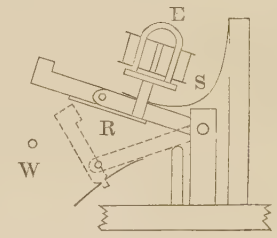


Fig. 6.

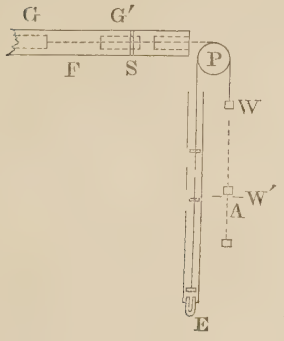


Fig. 7.

[1] Wied. Ann. **54**, 674, (1895).

a circular hole and it is placed at such a position that GG′=WW′, so that when the plate just reached the slit, the weight enters the hole. On this disc the two pieces of a metal are placed symmetrically about the centre along a certain diameter, and a narrow strip of tin foil was bridged over to connect them electrically, one end of which was pressed by a spring, while the other lightly resting on the metal electrode. This formed a part of the electric circuit of the electro-magnet in the hammer. Now if the circuit of the electro-magnet E in Fig. 7 was broken, the photographic plate began to move and at the instant when the plate came to the slit, the weight W fell down to A and pushed off the tin foil and broke the circuit of the electro-magnet attracting the hammer, and consequently the piano-wire was hammered and its subsequent motion was recorded by an optical means shown in Fig. 8. AB represents the stretched piano-wire, and F the moving plate camera. Two points *a*, *b* on wire were illuminated by an intense lights from electric arcs and their images were formed on the slit by lens, a needle point being placed just before the point *a* to give a reference line on the photograph. The hammer was adjusted to strike the point *b*, such that the image formed by the lens L_2 recorded the motion of the point *b* of the wire and head of the hammer. The same images of the point *a* and a needle, one above the image *b* and one below it, were formed by the lens L_1. The photographs shown in Fig. 9 and Fig. 10 were thus obtained. Four curves are seen in each photograph, the fine one in the middle part representing the vibration of the point struck by the hammer and the thick one the motion of the head of the hammer. Curves in the upper and the lower parts show the vibration of a certain point of the wire, so that a line connecting two corresponding points in these curves represents the position of the slit at same instant. The lines α and β were drawn parallel to the position of the slit on the photograph through the points at which the head of the hammer just touched on the wire and left it.

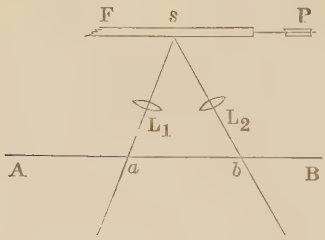


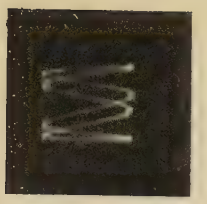
Fig. 8.

The intersections of the line α with these curves represent the state of the point *a* at the moment when the hammer just touched the wire and the corresponding point for the line β represents that of the same point when it lefts the wire.

Thus from the photographs we observe that the vibration of a point on the wire takes place as soon as the hammer lefts the wire, so that Kaufmann's assumption seems to be correct.

From the photograph taken, it is very easy to determine the instant at which a point of the wire enters into vibrations.

Fig. 3.



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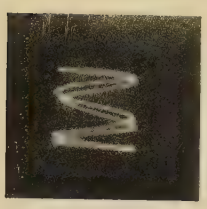

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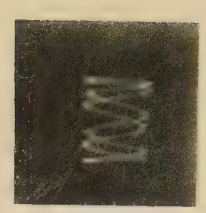

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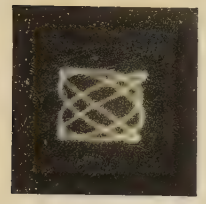

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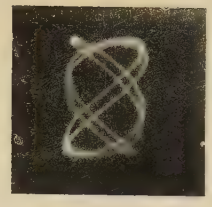

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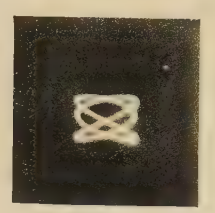

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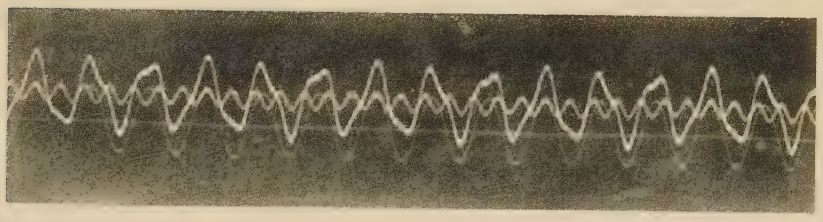
Fig. 5.



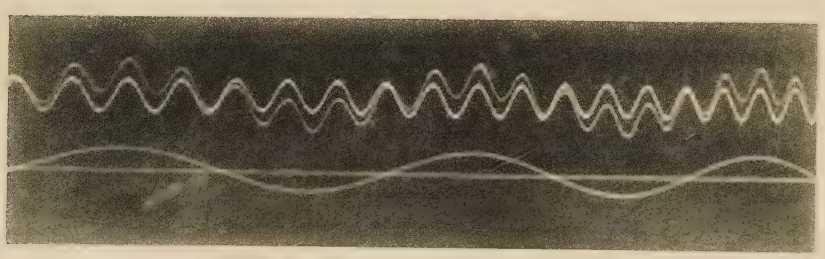


Fig. 9.

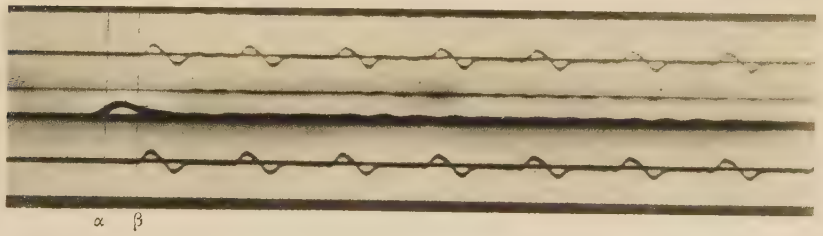
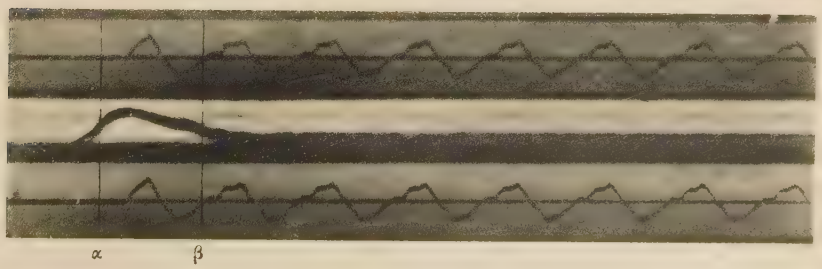


Fig. 10.



Demonstration of Steady Flow of Fluid.

By

Shuichi Hayashi

(Received March 1, 1922.)

The stream lines in two dimensional motion of viscous liquid were already studied experimentally by Hele-Shaw and others. The method used by them was to pass a series of streams of coloured liquid through a liquid contained between two pieces of glass plates separated at a small distance, the obstacle being placed between them. Lummer and Waetzmann[1] deviced an apparatus to demonstrate the forms of three dimentional flow of gases. But the photographs obtained by them were not so clear as those of Hele-Shaw. This is perhaps due to the mixing of air with tobacco-smoke at enterance into the wire gauze of their apparatus. The present writer, utilizing the method of Hele-Shaw, deviced an apparatus with which stream lines of running liquid as well as gaseous substance in the state of three dimensional motion were made clearly visible. The apparatus consists of a thin rectangular box with two rooms separated by a special bent partition as shown in Fig. 1. One of the flat sides is extended, and two series of small holes are pierced on this plate along the bottom of the box, each set belonging to the separate room, and the position of holes of the second series being thus little below the first one and coming between successive holes of the first series.

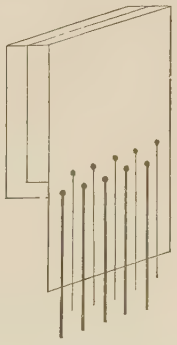


Fig. 1.

Two such boxes having exactly the same dimensions were prepared and they were placed

[1] Physik. Zs., **12**, 1135, (1911).

facing one another as shown by the dotted lines in Fig. 2, which shows the cross sections of an experimental apparatus used by the author. A liquid is led into the apparatus through the holes *a*, *b* and *d* simultaneously by rubber tubings from a reservoir, and a coloured liquid by the hole *c*, *e* being the outlet for the liquids. A wire gauze shown by double dotted lines is placed horizontally to render the flow uniform, and the flow is adjusted by the clips on the rubber tubings.

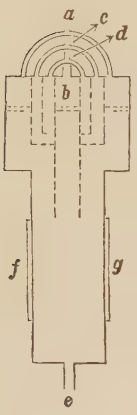


Fig. 2.

Now, when the water and a coloured water are sent through the apparatus the steady flow will soon be secured by adjusting the clips, and this may be observed through the glass window *fg*.

The same apparatus may be used for obtaining stream lines with gases. In this case, it is only necessary to open *a*, *b*, *d*, and connected *c* to a smoke chamber and *e* to a Bunsen water pump.

The writer also constructed a similar apparatus having circular cross section, and with these apparatus a good many interesting photographs of stream lines round various obstacles were taken. Fig. 3, Pl. I. illustrates stereoscopically the stream lines when a propellar is put in a steady running water in a circular tube, and Fig. 4 shows them when a sphere is put in a steady flow of air, while Figs. 5 and 6, Pl. II. represent the case when the sphere is uniformly rotated with different speeds.

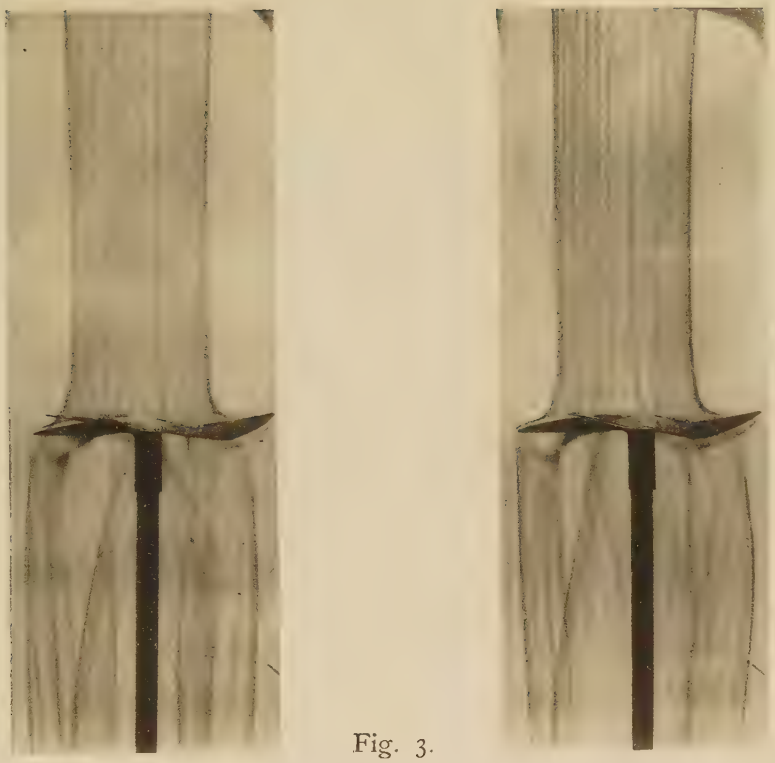


Fig. 3.

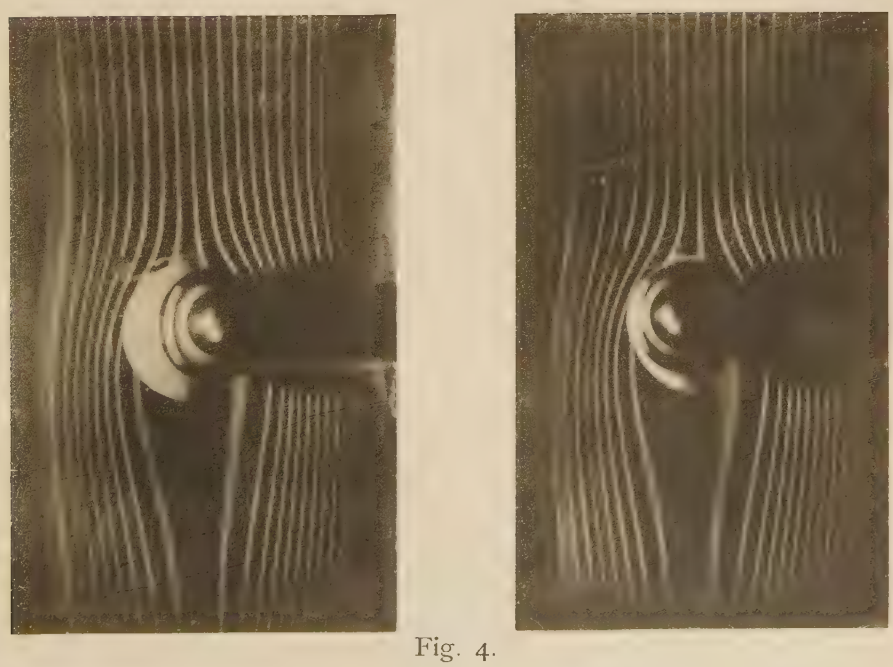


Fig. 4.

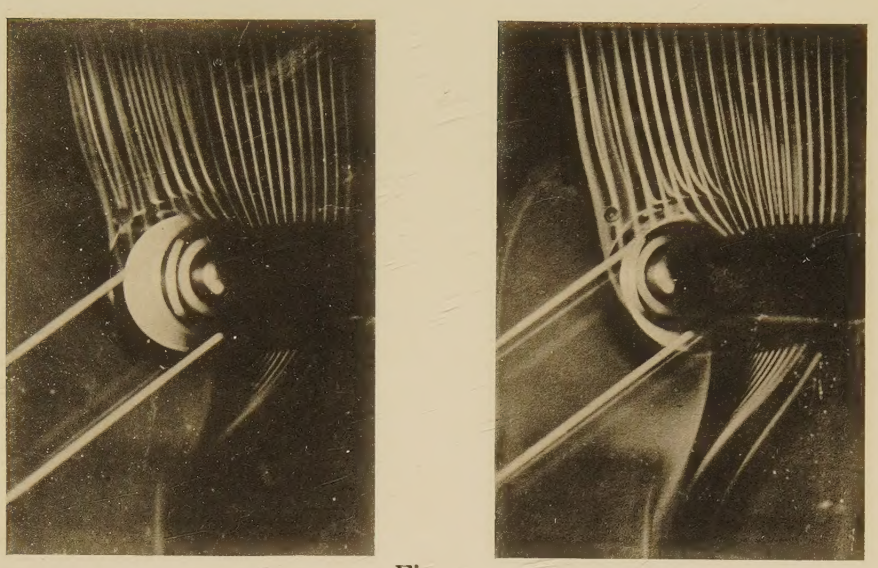


Fig. 5.

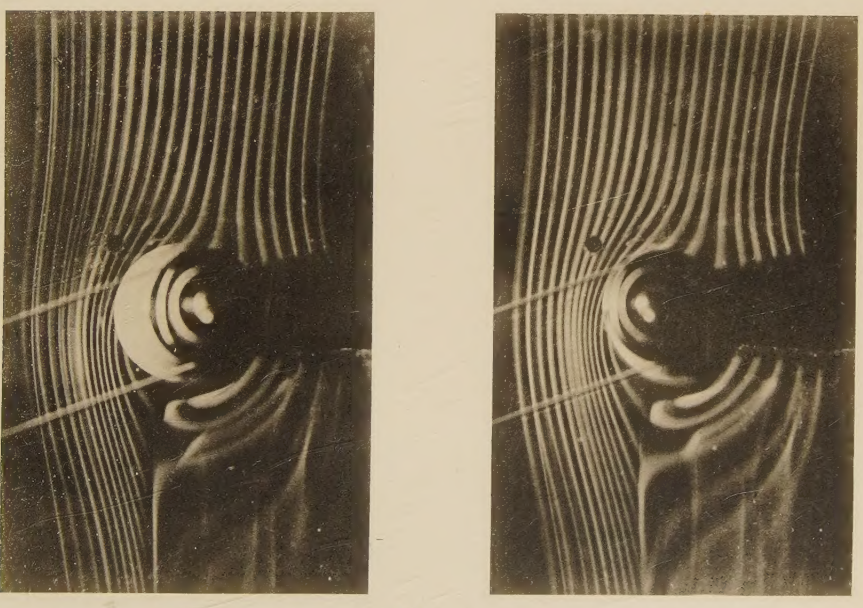


Fig. 6.

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